Source task and conventions

Approximate $y(t) = \int_0^t x(\tau)\,d\tau$ by constant-coefficient difference equations.

Conventional solution and proof

Let $t_n = nT$ and $y_0 = 0$. Since $y(t_n) - y(t_{n-1}) = \int_{t_{n-1}}^{t_n} x(\tau)\,d\tau$, left rectangles, right rectangles and trapezoids respectively give

$$y_n = y_{n-1} + Tx_{n-1}, \quad y_n = y_{n-1} + Tx_n, \quad y_n = y_{n-1} + \tfrac{T}{2}(x_n + x_{n-1}).$$

These have constant coefficients for a fixed T. If $|x'| \le M_1$, the per-interval rectangle error is at most $M_1T^2/2$, so at $t_N = NT$ the error is at most $NM_1T^2/2$. If $|x''| \le M_2$, subtracting the linear interpolant on one interval and integrating its interpolation remainder gives trapezoidal error at most $M_2T^3/12$, hence total at most $NM_2T^3/12$.