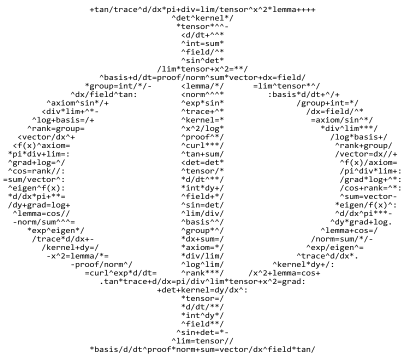
PHI TECHNICAL MONOGRAPHS

Volume X

Potential Theory and Harmonic Geometry

Harmonic analysis, boundary problems and energy methods



PHI research collective

Scope and reading routes

Potential theory studies how local averaging, boundary geometry and global constraints determine a field. In the plane its central equations are

$$\Delta u = 0, \qquad \Delta v \geq 0, \qquad -\Delta G(\,\cdot\,, p) = 2\pi\delta_p.$$

The first describes harmonic equilibrium, the second subharmonic comparison, and the third a normalized point source. Their common structure connects complex differentiation to boundary-value problems, measures, conformal geometry and energy minimization.

This volume continues the PHI Technical Monographs through Section 10, "Appendix 1: Potential theory," of Mikhail Lyubich's *Conformal Geometry and Dynamics of Quadratic Polynomials*, volumes I–II. It develops the source material into a sustained mathematical account: local differential identities lead to Poisson representation and Harnack compactness; subharmonic comparison leads to Perron envelopes and regular boundary values; logarithmic potentials and harmonic measure lead to Green functions, critical topology, electrostatic models and controlled numerical approximation. The source correspondence chapter records the precise coverage and the corrections required by the supplied source's formulations.

The principal reading route follows the chapters in order. A reader already comfortable with harmonic functions may begin with subharmonic comparison, returning to the early chapters for conventions and estimates. A boundary geometry route joins the Poisson, Perron, barrier, harmonic-measure and Green chapters. An applications route joins the spectral, energy, electrostatic and numerical chapters; the hypotheses established in the intervening analysis remain part of each result.

Every authored investigation has a worked solution. The diagrams represent explicit analytic models or labeled geometric constructions. Their sampled curves illustrate the arguments; they are not experimental measurements or proofs of assertions between sample points. Monochrome line styles and monospace labels support reading in both light and dark presentations.

The text assumes undergraduate real and complex analysis, elementary integration and the basic topology of plane domains. It declares more substantial inputs where used, notably positive-functional measure representation, Jordan–Schoenflies topology, the Riemann mapping theorem and boundary extension on Jordan domains. Written proofs in this volume are relative to those named inputs. A general nonregular-domain capacity criterion is discussed as a remaining extension, rather than inferred from the regular-domain Green theory proved here.

The public and internal editions share the mathematical core. The internal edition adds teaching routes and assessment material. Both presentations use a single current manuscript. The accompanying editable-source archive records statements, source provenance and production checks. Those checks establish file integrity and presentation properties; no new proof-assistant, CAS, external peer-review, physical-measurement or hardware-validation claim is made. Formalization follows the content-first stage separately.

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Chapter 1

Harmonic functions, differentials and periods

Potential theory begins with a local balance equation and ends with global information about a domain. The same harmonic function can describe a temperature, an electrostatic potential, a conformal coordinate or a boundary averaging rule. These interpretations use different units and different boundary data. Before using any of them, we fix the differential conventions that determine the signs, normalizations and possible periods of the mathematical problem.

Throughout the planar chapters, a domain is a nonempty connected open subset $U \subset \mathbb{C}$. We identify $z = x + iy$ with $(x, y) \in \mathbb{R}^2$, use the orientation $dx \wedge dy$, and write

$$dA = dx\,dy, \qquad \Delta = \partial_x^2 + \partial_y^2, \qquad \mathcal{H}(U) = \{u \in C^2(U;\mathbb{R}) : \Delta u = 0\}.$$

The notation $K \Subset U$ means that K is compact and contained in U. For a domain on a Riemann surface, harmonicity is defined in every conformal coordinate; the coordinate calculation below proves consistency. Boundary continuity is a separate assumption, denoted explicitly by $u \in \mathcal{H}(U) \cap C(\overline{U})$. It is never inferred merely from $u \in \mathcal{H}(U)$.

1.1 The rotation convention and local conjugates

For a real one-form $\omega = a\,dx + b\,dy$ define

$$*\omega = -b\,dx + a\,dy, \qquad *^2\omega = -\omega, \qquad d^c u = *du = -u_y\,dx + u_x\,dy.$$

Thus $*dx = dy$ and $*dy = -dx$. Exterior differentiation gives

$$\begin{aligned} dd^c u &= d(-u_y\,dx + u_x\,dy) \\ &= u_{yy}\,dx \wedge dy + u_{xx}\,dx \wedge dy = (\Delta u)\,dx \wedge dy. \end{aligned}$$

The order of x, y in the last area form matters. The source's displayed $dz \wedge dy$ at this point is replaced by the real area form used in the definition. With $\partial u = \frac{1}{2}(u_x - iu_y)\,dz$ and $\overline{\partial} u = \frac{1}{2}(u_x + iu_y)\,d\bar{z}$, one also has

$$\partial = \tfrac{1}{2}(d + id^c), \qquad \overline{\partial} = \tfrac{1}{2}(d - id^c), \qquad dd^c u = 2i\partial\overline{\partial} u.$$

These identities act on functions in the displayed convention. A different normalization of d^c changes both the Laplacian measure and the Green-function constant; we will keep this convention throughout the volume.

Proposition 1.1 (Local harmonic conjugacy)**.** *Every $u \in \mathcal{H}(U)$ is locally the real part of a holomorphic function. On a connected region where a conjugate exists, it is unique up to a real constant. Conversely, the real and imaginary parts of a holomorphic function are harmonic.*

Proof. Choose a rectangle $Q \Subset U$ containing (x_0, y_0). Set

$$v(x,y) = -\int_{x_0}^{x} u_y(t, y_0)\, dt + \int_{y_0}^{y} u_x(x, s)\, ds.$$

Differentiating the second integral with respect to y gives $v_y = u_x$. Differentiation with respect to x, followed by $u_{xx} = -u_{yy}$, gives

$$\begin{aligned} v_x(x,y) &= -u_y(x, y_0) + \int_{y_0}^{y} u_{xx}(x, s)\, ds \\ &= -u_y(x, y_0) - \int_{y_0}^{y} u_{yy}(x, s)\, ds = -u_y(x, y). \end{aligned}$$

Consequently $dv = d^c u$ and $u + iv$ satisfies the Cauchy–Riemann equations. Its first partial derivatives are continuous, so the complex derivative exists:

$$(u + iv)' = u_x + i v_x = u_x - i u_y.$$

To spell out this last implication, real differentiability at a point a gives, for $h = \xi + i\eta$,

$$F(a + h) - F(a) = F_x(a)\xi + F_y(a)\eta + o(|h|).$$

The equations just proved say $F_y = iF_x$. Hence the linear part is $F_x(a)(\xi + i\eta) = F_x(a)h$, and division by $h \neq 0$ leaves a remainder whose modulus tends to zero independently of the direction of approach. Thus the asserted complex derivative follows from a genuine two-variable remainder estimate, not just agreement of two directional derivatives.

If v_1, v_2 are conjugates, then $d(v_1 - v_2) = 0$. The difference is constant on each small rectangle, and hence on the connected domain. Conversely, a holomorphic function has a convergent local power series. Its real and imaginary parts are smooth, and the Cauchy–Riemann equations imply $u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$ and the analogous identity for v. □

This proof also establishes that every C^2 harmonic function is smooth and real analytic. No higher differentiability was assumed in the initial equation. Writing a local holomorphic series $F(z) = \sum_{n \geq 0} c_n (z - a)^n$ gives

$$u(z) = \operatorname{Re} F(z), \qquad 2\partial u = F'(z)\, dz.$$

The local holomorphic functions may differ by imaginary constants, but the one-form $2\partial u$ is globally well defined. This distinction is exactly what allows a harmonic function to have a multivalued conjugate.

1.2 Flux, monodromy and the missing radius

Let $\gamma(s) = (x(s), y(s))$ be parametrized by arclength. Its unit tangent is $T = (x', y')$ and its right-hand normal is $n = (y', -x')$. Then

$$d^c u(T) = -u_y x' + u_x y' = \nabla u \cdot n = \partial_n u.$$

For the positively oriented boundary of a planar region the right-hand normal is the outward normal. Green's formula is therefore consistent with

$$\int_{\partial\Omega} d^c u = \int_{\partial\Omega} \partial_n u\, ds = \int_{\Omega} \Delta u\, dA.$$

Here Ω is taken with sufficiently regular boundary for the usual multivariable-calculus divergence theorem; later singular identities will be proved by deleting a small disk and passing to a limit.

Theorem 1.2 (The period obstruction)**.** *For $u \in \mathcal{H}(U)$, the following are equivalent:*

(i) there is a globally defined v with $u + iv$ holomorphic on U;
(ii) $\int_\gamma d^c u = 0$ for every closed piecewise smooth path γ in U;
(iii) the value $\int_{z_0}^{z} d^c u$ is independent of the path from z_0 to z.

For a positively oriented circle $z = a + re^{i\theta}$, the period equals

$$\int_{|z-a|=r} d^c u = \int_0^{2\pi} r\, \partial_r u(a + re^{i\theta})\, d\theta.$$

Proof. If $dv = d^c u$, integration along a closed path is zero by the fundamental theorem for line integrals. Conversely, two paths with the same endpoints, traversed successively in opposite directions, form a closed path. Vanishing of its integral proves path independence. Define $v(z)$ by any such path. Inside a small rectangle, append a short coordinate segment to a fixed path; the one-variable fundamental theorem then gives $v_x = -u_y$ and $v_y = u_x$. The local calculation proves that v is a conjugate on all of U. On the circle, the outward normal is radial and $ds = r\, d\theta$, proving the displayed formula. Omitting r would give an expression with the wrong dependence on radius even for $u(z) = \log|z - a|$. □

Because $dd^c u = 0$, a smooth homotopy of paths with fixed endpoints preserves the integral: apply Stokes' theorem to the parameter rectangle, whose two remaining edges are constant. Subdivision into coordinate rectangles proves the same statement for piecewise smooth homotopies. On a simply connected domain all closed paths are null-homotopic, so all periods vanish. More generally, periods factor through the first homology group. We need only the concrete implication that integrals over boundaries of finitely many oriented pieces cancel along shared edges and vanish when $dd^c u = 0$.

For $u(z) = \log|z|$ on $\mathbb{C} \setminus \{0\}$,

$$du = \frac{x\,dx + y\,dy}{x^2 + y^2}, \qquad d^c u = \frac{-y\,dx + x\,dy}{x^2 + y^2}, \qquad \int_{|z|=r} d^c u = 2\pi.$$

The local conjugate is the argument, whose increment on one positive circuit is 2π. Thus $\log|z|$ is globally harmonic while its conjugate is not single valued. The analytic continuation of $u + iv$ gains $2\pi i$.

1.2.1 Several punctures: a complete period calculation

Let $a_1, \dots, a_m$ be distinct points in a disk D, let $U = D \setminus \{a_1, \dots, a_m\}$, and suppose

$$u(z) = H(z) + \sum_{j=1}^{m} \alpha_j \log|z - a_j|, \qquad \alpha_j \in \mathbb{R}, \quad H \in \mathcal{H}(D).$$

The conjugate of H is single valued because the full disk is simply connected. For a closed piecewise smooth curve $\gamma \subset U$, direct integration therefore gives

$$\begin{aligned} \int_\gamma d^c u &= \sum_{j=1}^{m} \alpha_j \int_\gamma d\arg(z - a_j) \\ &= 2\pi \sum_{j=1}^{m} \alpha_j \operatorname{Ind}(\gamma, a_j). \end{aligned}$$

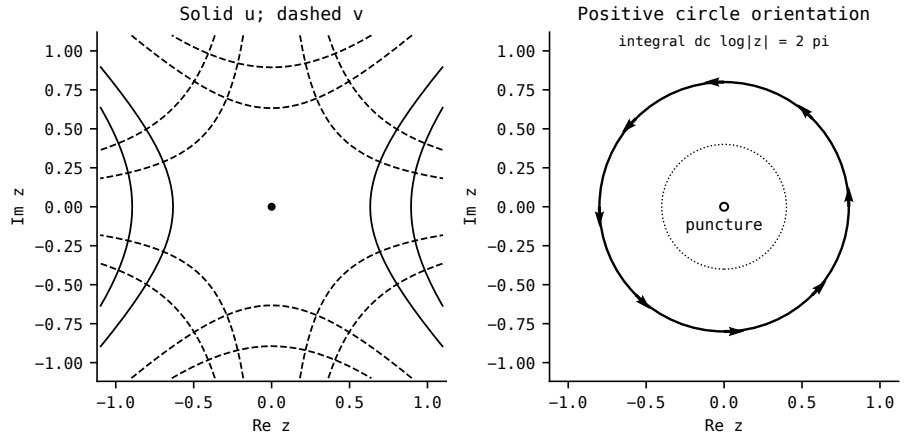


Figure 1.1: Level curves of $u = x^2 - y^2$ and its conjugate $v = 2xy$ meet orthogonally away from the origin. The second panel records the nonzero period of $d^c \log|z|$ around a puncture. These are exact analytic models; the drawn curves do not replace the period calculation.

Here the index is an integer: a continuous argument along the parameter interval changes by an integer multiple of 2π when the curve closes. Equivalently it is $(2\pi i)^{-1} \int_\gamma dz/(z - a_j)$. This computation also explains why deforming the curve without crossing a puncture preserves the period. The integrands are closed away from the punctures, and opposite sides of each subdividing region cancel.

The small circles around the individual punctures have index vectors equal to the coordinate basis vectors of $\mathbb{Z}^m$. Consequently u has a global harmonic conjugate if and only if every $\alpha_j = 0$. A cancellation on one outer circle is insufficient: that circle sees only the sum $\sum \alpha_j$. For example, $\log|z - a_1| - \log|z - a_2|$ has zero period around a loop enclosing both points but periods 2π and -2π around the two separate points. This is an explicit distinction between one tested flux and vanishing of every period.

There is a weaker condition under which the exponential of a local holomorphic potential is single valued. Its continuation around γ multiplies it by

$$\exp\left(2\pi i \sum_j \alpha_j \operatorname{Ind}(\gamma, a_j)\right).$$

The multiplier is one for every loop exactly when all $\alpha_j \in \mathbb{Z}$. In that case, if K is a holomorphic function with real part H,

$$e^{u+iv} = Ce^{K(z)} \prod_{j=1}^{m} (z - a_j)^{\alpha_j}, \qquad |C| = 1.$$

Negative integer powers give poles at the removed points; positive powers give zeros. There is no contradiction with the lack of a single-valued logarithm: exponentiation has precisely erased the integral multiples of $2\pi i$. More generally, the periods on selected loops $\gamma_1, \ldots, \gamma_\ell$ form the finite linear system

$$\mathbf{p} = 2\pi W \boldsymbol{\alpha}, \qquad W_{kj} = \operatorname{Ind}(\gamma_k, a_j) \in \mathbb{Z}.$$

Its nullspace describes which logarithmic coefficients the chosen loops fail to detect. This matrix records topology, rather than a numerical approximation to it.

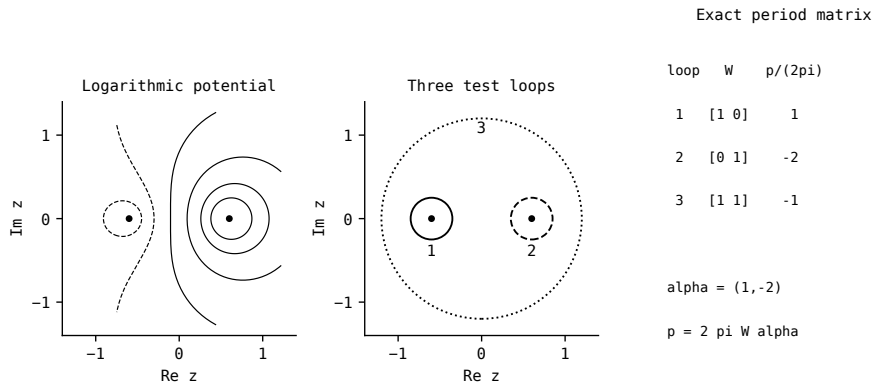


Figure 1.2: Two punctures with logarithmic weights 1 and -2. The left panel shows potential contours with the singularities masked. The middle panel displays separate and enclosing loops. The right panel records their index vectors and exact periods: 2π, -4π, and -2π. The finite period matrix exposes information lost by testing only an outer loop.

1.3 Coordinate changes and harmonic forms

Proposition 1.3 (Conformal covariance)**.** *If $\phi : V \to U$ is holomorphic and $u \in C^2(U)$, then*

$$\Delta(u \circ \phi)(z) = |\phi'(z)|^2 (\Delta u)(\phi(z)).$$

In particular, a harmonic function pulls back to a harmonic function, even at a critical point of ϕ. For conformal coordinate changes this defines harmonicity consistently on a Riemann surface.

Proof. Write $\phi = (p, q)$. The chain rule gives

$$\begin{aligned}\Delta(u \circ \phi) = u_{xx}(|\nabla p|^2) + 2u_{xy}(\nabla p \cdot \nabla q) + u_{yy}(|\nabla q|^2) \\ + u_x \Delta p + u_y \Delta q,\end{aligned}$$

where derivatives of u are evaluated at (p, q). The Cauchy–Riemann identities give $\Delta p = \Delta q = 0$, $\nabla p \cdot \nabla q = 0$ and $|\nabla p|^2 = |\nabla q|^2 = |\phi'|^2$. Substitution proves the formula, including points where all first derivatives of ϕ vanish. Alternatively, a local representation $u = \operatorname{Re} F$ gives $u \circ \phi = \operatorname{Re}(F \circ \phi)$ directly. The first route records the covariance factor; the second explains why no injectivity is needed for the pullback assertion. □

If a conformal metric is $ds^2 = \lambda^2(dx^2 + dy^2)$ with $\lambda > 0$, then $\Delta_g u = \lambda^{-2} \Delta u$. Its zero set as an equation is therefore independent of the chosen conformal scale. The energy has the same cancellation:

$$|\nabla_g u|_g^2 \, dA_g = \lambda^{-2} |\nabla u|^2 \lambda^2 \, dA = |\nabla u|^2 \, dA.$$

This is a specifically two-dimensional identity; the later energy chapter will use it with explicit boundary conditions rather than as a dimensional analogy for arbitrary physical space.

A C^1 real one-form $\omega = a\,dx + b\,dy$ is locally a differential of a harmonic function exactly when it is both closed and co-closed. Indeed,

$$d\omega = (b_x - a_y)\, dx \wedge dy, \qquad d(*\omega) = (a_x + b_y)\, dx \wedge dy.$$

The equations $b_x = a_y$ and $a_x = -b_y$ are the Cauchy–Riemann equations for $a - ib$. Hence

$$\omega + i * \omega = (a - ib)\, dz$$

is holomorphic. Conversely, the same equations imply local exactness of ω, by the rectangle construction already used, and the potential has Laplacian $a_x + b_y = 0$. The form itself is globally defined even when its potential has periods. We will distinguish a closed form, an exact form and the differential of a single-valued harmonic function whenever topology enters.

1.4 Worked investigations

Investigation 1.4 (A sign audit)**.** For $u = x^2 - y^2$, compute $du, d^c u, dd^c u$ and a harmonic conjugate. Determine the clockwise flux through the unit circle.

Solution. One obtains $du = 2x\,dx - 2y\,dy$ and $d^c u = 2y\,dx + 2x\,dy = d(2xy)$, so $dd^c u = 0$ and $v = 2xy$. Both orientations have zero total period. Directly on $r = 1$, $\partial_r u = 2\cos 2\theta$, and $\int_0^{2\pi} 2\cos 2\theta\,d\theta = 0$. Changing orientation changes the sign of the integrand's flux convention but not this zero value.

Investigation 1.5 (When an exponential is single valued)**.** Suppose u is harmonic and its local conjugates have periods in $2\pi\mathbb{Z}$. Show that e^{u+iv} can be defined globally after choosing one phase. What fails if one period equals π?

Solution. Analytic continuation around a loop changes $u + iv$ by iP. The exponential gains e^{iP}, which is one exactly when $P \in 2\pi\mathbb{Z}$. Choosing the value at a basepoint fixes the multiplicative phase and makes continuation independent of path. A period π changes the sign; the local exponentials then cannot be glued into a nonzero single-valued function. For $u = \alpha \log|z|$, the condition is precisely $\alpha \in \mathbb{Z}$.

Investigation 1.6 (A branched pullback)**.** Let $u(w) = \operatorname{Re} w$ and $\phi(z) = z^3$. Verify harmonicity at the critical point 0, and compare with the pullback of $|w|^2$.

Solution. The first pullback is $x^3 - 3xy^2$, whose Laplacian is zero. For the second, $|\phi(z)|^2 = |z|^6$, so $\Delta|z|^6 = 36|z|^4$. The covariance formula gives exactly $|3z^2|^2 \cdot 4 = 36|z|^4$. The Laplacian can vanish at a critical point without vanishing in a neighborhood; pointwise cancellation alone does not make a function harmonic.

Investigation 1.7 (Closed need not mean exact)**.** Show that $\omega = (-y\,dx + x\,dy)/(x^2 + y^2)$ is closed and co-closed on a round annulus, but not exact there.

Solution. It is $d^c \log|z|$, so it is closed. Also $*\omega = -d\log|z|$, so it is co-closed. Its integral on a circle is 2π, whereas an exact form integrates to zero on every closed path. Locally it is $d\theta$; the failure concerns a global choice of θ.

Investigation 1.8 (Metric scale and flux)**.** For the metric $\lambda^2|dz|^2$, show that the boundary quantity $\partial_{n_g} u\,ds_g$ equals $\partial_n u\,ds$.

Solution. Unit normal vectors scale by λ^{-1} while arclength scales by λ. Thus $\partial_{n_g} u = \lambda^{-1}\partial_n u$ and $ds_g = \lambda ds$. The factors cancel. This makes the period/flux identity conformally natural, even though each separate factor depends on the metric.

Chapter 2

Mean values, maximum principles and uniqueness

2.1 Circle means and two routes to the mean-value property

For $\overline{D(a,r)} \subset U$ define

$$M_u(a,r) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})\, d\theta.$$

The closed-disk containment gives access to the complete circle and to a neighborhood of it. When only $D(a,R) \subset U$ is given, use $r < R$.

Theorem 2.1 (Mean values and the differential equation)**.** *A C^2 function u is harmonic exactly when $M_u(a,r) = u(a)$ for every closed disk contained in its domain. A harmonic function also has the disk mean-value property*

$$u(a) = \frac{1}{\pi r^2}\int_{D(a,r)} u\, dA.$$

Proof. For the first forward proof, take a holomorphic conjugate on the disk and apply the Cauchy integral formula to its center. Taking real parts gives the circle mean. For a second proof that retains the PDE interpretation, differentiate the mean with respect to radius:

$$\frac{d}{dr}M_u(a,r) = \frac{1}{2\pi r}\int_{\partial D(a,r)} \partial_n u\, ds = \frac{1}{2\pi r}\int_{D(a,r)} \Delta u\, dA.$$

If $\Delta u = 0$, the mean is independent of r; continuity at $r = 0$ identifies the constant as $u(a)$. Integrating the circle identity against $2s\, ds/r^2$ for $0 < s < r$ gives the disk mean.

Conversely, Taylor's formula at a reads

$$u(a+h) = u(a) + \nabla u(a)\cdot h + \tfrac{1}{2}h^T D^2 u(a) h + o(|h|^2).$$

The remainder is uniform in the direction of h because the Hessian is continuous. On $h = r(\cos\theta, \sin\theta)$ the linear terms and the mixed quadratic term integrate to zero, while each square has mean $r^2/2$. Therefore

$$M_u(a,r) - u(a) = \frac{r^2}{4}\Delta u(a) + o(r^2).$$

If the left side is zero for all small r, divide by r^2 and let r decrease to zero. This yields $\Delta u(a) = 0$ at every point.

The uniform remainder can be bounded explicitly. With the operator norm of matrices, set

$$\omega_a(r) = \sup_{|h|\le r} \|D^2u(a+h) - D^2u(a)\|.$$

Taylor's integral remainder has the form

$$R_a(h) = \int_0^1 (1-t)\, h^T \big(D^2u(a+th) - D^2u(a)\big) h \, dt, \qquad |R_a(h)| \le \tfrac{1}{2}\omega_a(r)|h|^2.$$

Thus the averaging calculation proves the quantitative estimate

$$\left| \frac{4}{r^2}\big(M_u(a,r) - u(a)\big) - \Delta u(a) \right| \le 2\omega_a(r) \longrightarrow 0.$$

This supplies the precise limiting argument behind division by r^2.

□

The equality is more rigid than an approximate average. For example, $u(z) = |z|^2$ satisfies $M_u(a,r) - u(a) = r^2$, not zero. An error estimate $|M_u(a,r) - u(a)| \le Cr^2$ bounds the Laplacian by $4C$ when the Taylor argument applies; it does not imply harmonicity. An error $o(r^2)$ does.

Lemma 2.2 (Continuous mean-value functions are smooth)**.** *If u is continuous and has the circle mean-value property on every small disk, then u is smooth and harmonic.*

Proof. Choose a nonnegative smooth radial function η supported in the unit disk with integral one, and put $\eta_\varepsilon(z) = \varepsilon^{-2}\eta(z/\varepsilon)$. On points whose distance to the boundary exceeds ε, polar coordinates give

$$\begin{aligned}(u * \eta_\varepsilon)(a) &= \int_0^\varepsilon \eta_\varepsilon(r) \left(\int_0^{2\pi} u(a - re^{i\theta})\, d\theta \right) r\, dr \\ &= u(a) \int_{\mathbb{C}} \eta_\varepsilon \, dA = u(a).\end{aligned}$$

On a fixed compact neighborhood we may differentiate the convolution under the integral by differentiating its smooth compactly supported kernel. Continuity bounds u on the relevant compact set, so each derivative is justified by a bounded integrable majorant. Thus $u = u * \eta_\varepsilon$ is smooth there. The converse part of Theorem 2.1 now applies.

For clarity, differentiating is local in the center variable. Choose $K \Subset V \Subset U$ and $\varepsilon < \operatorname{dist}(K, \partial V)$. Extend $u|_V$ by zero outside V only for the purpose of writing its convolution near K. No boundary regularity of that extension is needed, because the kernel centered at $a \in K$ sees only points of V. For a multi-index β the derivative is

$$D^\beta(u * \eta_\varepsilon)(a) = \int_V u(w) D^\beta \eta_\varepsilon(a - w)\, dA(w).$$

The integrand is bounded by $\sup_{\overline{V}} |u|\, \varepsilon^{-2-|\beta|} \|D^\beta \eta\|_\infty$ on a fixed compact support, after choosing V with compact closure. Dominated convergence proves every differentiation. One fixed positive ε works throughout a neighborhood of K; there is no limiting interchange of arbitrarily high derivatives.

□

2.2 Global maxima, local maxima and boundary limits

Theorem 2.3 (Maximum principles)**.** *Let $u \in \mathcal{H}(U)$.*

(i) If u attains its global supremum or infimum at an interior point, then it is constant.
(ii) If u has a local maximum or minimum at an interior point, it is constant.
(iii) If $\overline{U}$ is compact, $\partial U \neq \varnothing$ and $u \in C(\overline{U})$, then $\min_{\partial U} u \leq u(z) \leq \max_{\partial U} u$ for $z \in U$.

The statements also hold in conformal coordinates on a connected Riemann surface.

Proof. For (i), let $M = u(a) = \sup_U u$. If any point on a sufficiently small circle about a had value below M, continuity would give an arc on which the value is bounded below M by a fixed positive amount. The average would then be less than M, contrary to the mean-value property. Every such circle, and hence the entire small disk, has value M. The set $\{u = M\}$ is consequently open as well as closed; connectedness makes it all of U. Apply this argument to $-u$ for the minimum.

For (ii), the local maximum is a global maximum on some small disk, so (i) makes u constant on that disk. The local holomorphic one-form $2\partial u$ vanishes on a nonempty open set. The identity theorem propagates its vanishing through overlapping connected coordinate neighborhoods: a holomorphic coefficient that is not identically zero has isolated zeros by its first nonzero power-series term. Thus $du = 0$ throughout the connected domain and u is constant. This unique-continuation step is special to harmonic functions; it cannot be reused for arbitrary subharmonic functions.

For (iii), continuity and compactness give a maximum on $\overline{U}$. If it is attained in U, (i) makes the function constant, so the same value is attained on the nonempty boundary. Otherwise it is attained on the boundary already. Apply the same reasoning to the minimum. Coordinate disks provide exactly the same local argument on a Riemann surface. □

The compactness and nonempty-boundary hypotheses are separate. A compact surface without boundary has only constant harmonic functions, but the expression $\max_{\partial U} u$ has no meaning when ∂U is empty. Likewise, the function $u(x + iy) = y$ on the upper half-plane is zero on its finite boundary yet positive inside. Behavior at infinity must be controlled before a boundary uniqueness statement is made on an unbounded domain.

Proposition 2.4 (Boundary limsup form)**.** *Let U have compact closure and nonempty boundary. If $u \in \mathcal{H}(U)$ and $\limsup_{z \to \xi,\, z \in U} u(z) \leq M$ at every $\xi \in \partial U$, then $u \leq M$ in U. In particular, two harmonic functions with the same continuous boundary values coincide.*

Proof. Fix $\varepsilon > 0$. At each boundary point there is a neighborhood in which $u < M + \varepsilon$ on U. Finitely many cover the compact boundary. Their complement in $\overline{U}$ is a compact subset of U. If $u(z_0) > M + \varepsilon$, the supremum of u is attained on that compact subset and is a global interior maximum. The preceding theorem forces constancy, contrary to the boundary upper bound. Hence $u \leq M + \varepsilon$; let $\varepsilon \downarrow 0$. For uniqueness apply this conclusion to both $u - v$ and $v - u$. □

2.3 A comparison estimate with a nonzero residual

Maximum principles also turn a PDE residual into an error bound, provided one supplies a comparison function. This is an analytic estimate, not a claim that a small sampled residual certifies a numerical solution.

Proposition 2.5 (Residual bound in a disk)**.** *Let $w \in C^2(D(a,R)) \cap C(\overline{D(a,R)})$ satisfy $|\Delta w| \leq \delta$ everywhere and $|w| \leq \varepsilon$ on the boundary. Then*

$$|w(z)| \leq \varepsilon + \frac{\delta}{4}(R^2 - |z-a|^2).$$

Proof. Put $b(z) = \varepsilon + \delta(R^2 - |z-a|^2)/4$. Then $\Delta b = -\delta$, so $\Delta(w-b) \geq 0$ and $w - b \leq 0$ on the boundary. For a C^2 function q with $\Delta q \geq 0$, add $\tau|z-a|^2$ with $\tau > 0$. The result has strictly positive Laplacian and cannot have an interior maximum, because the Hessian at such a maximum would be negative semidefinite and have nonpositive trace. Its maximum is therefore on the boundary. Let $\tau \downarrow 0$ to obtain the weak maximum principle for q. Apply it to $w - b$ and to $-w - b$. □

The constant $1/4$ is exact: $w(z) = \delta(R^2 - |z-a|^2)/4$ has boundary zero and $|\Delta w| = \delta$. The estimate separates boundary error from interior forcing. A residual given only at finitely many points does not satisfy the hypothesis $|\Delta w(z)| \leq \delta$ for every z.

2.4 Worked investigations

Investigation 2.6 (Recovering a Laplacian from averages)**.** Compute $M_u(0,r)$ for $u(x,y) = 3x^2 + 2xy - y^2$ and verify the second-order mean formula. Is u harmonic?

Solution. The means of x^2, y^2 are $r^2/2$ and that of xy is zero, so $M_u(0,r) = r^2$. Also $\Delta u = 6 - 2 = 4$. Thus $M_u(0,r) - u(0) = r^2\Delta u/4 = r^2$, and the function is not harmonic.

Investigation 2.7 (A disconnected counterexample)**.** Explain why connectedness is necessary for an interior maximum to force constancy throughout the set.

Solution. On two disjoint disks take $u = 1$ on the first and $u = 0$ on the second. It is harmonic on their union and attains its global maximum at every point of the first disk, but is not constant on the union. The theorem applies to each connected component separately.

Investigation 2.8 (An unbounded domain)**.** Find a nonzero harmonic function on the upper half-plane that is continuous and zero on its finite boundary. Identify the missing hypothesis in the bounded-domain uniqueness proof.

Solution. The function $u(x+iy) = y$ works. The compact subset on which a putative positive supremum must be attained no longer exists: values can increase along iy as $y \to \infty$. A growth condition or an explicit boundary condition at infinity is required.

Investigation 2.9 (Polynomial decomposition)**.** Find the harmonic function in the unit disk with boundary values x^2. Express x^2 as that harmonic function plus a zero-boundary correction.

Solution. On the circle $x^2 = (1 + \cos 2\theta)/2$. The harmonic extension is $h(x,y) = (1 + x^2 - y^2)/2$. Then $x^2 - h = (x^2 + y^2 - 1)/2$, which vanishes on the circle and has Laplacian 2. The equality also follows by checking these two properties and using uniqueness. It exhibits the distinction between extending the boundary polynomial and using the same polynomial in the interior.

Investigation 2.10 (Uniform limits)**.** Suppose harmonic functions converge uniformly on compact subsets to u. Prove harmonicity of u without differentiating the sequence.

Solution. Compact-uniform convergence preserves continuity. Fix a closed disk inside the domain; convergence on its circle and at its center allows the limit to pass through the finite-length integral. Thus u has the circle mean property. Lemma 2.2 supplies smoothness and the equation. Uniform convergence of derivatives was not an input.

Investigation 2.11 (A sharp forcing estimate)**.** If $\Delta w = 8$ on a disk of radius 3 and $w = 0$ on its boundary, determine w and its maximum absolute value.

Solution. The polynomial $w(z) = 2(|z-a|^2 - 9)$ has the prescribed Laplacian and boundary values. Its difference from any other solution is harmonic with zero boundary values, so it is the unique solution. Its absolute maximum is 18, attained at the center, agreeing with $\delta R^2/4 = 8 \cdot 9/4$.

2.4.1 Higher-order averaging and a shaped residual barrier

For a C^4 function, one more Taylor order makes the average calculation more informative. The required fourth moments are

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^4\theta\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\sin^4\theta\,d\theta = \frac{3}{8}, \qquad \frac{1}{2\pi}\int_0^{2\pi}\cos^2\theta\sin^2\theta\,d\theta = \frac{1}{8}.$$

Odd moments vanish under a half-turn or reflection. Substituting the fourth-order Taylor polynomial and collecting its mixed terms gives

$$M_u(a,r) = u(a) + \frac{r^2}{4}\Delta u(a) + \frac{r^4}{64}\Delta^2 u(a) + o(r^4).$$

Indeed the coefficient of u_{xxyy} is $6(1/24)(1/8) = 1/32$, twice the coefficient $1/64$ of each pure fourth derivative. For the diagnostic $A_r = 4(M_u(a,r) - u(a))/r^2$, this yields

$$A_r = \Delta u(a) + \frac{r^2}{16}\Delta^2 u(a) + o(r^2), \qquad \frac{4A_{r/2} - A_r}{3} = \Delta u(a) + o(r^2).$$

The cancellation is an exact algebraic statement about the leading error. It does not control roundoff or an independently approximated integral. If an average is supplied with absolute error at most ϵ_r, its contribution to A_r may be as large as $4\epsilon_r/r^2$. Decreasing the radius therefore reduces the analytic bias while amplifying a fixed measurement or quadrature error. The two effects have different origins.

A comparison barrier can also follow the shape of the domain. On the ellipse

$$E = \{(x,y) : x^2/a^2 + y^2/b^2 < 1\}, \qquad a, b > 0,$$

suppose $|\Delta w| \le \delta$ and $|w| \le \epsilon$ on the boundary. Here $w \in C^2(E) \cap C(\overline{E})$ and $\epsilon, \delta \ge 0$. Define

$$B(x,y) = \epsilon + \frac{\delta}{2(a^{-2} + b^{-2})}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right).$$

Since $\Delta B = -\delta$, the same two comparisons as in Proposition 2.5 show $-B \le w \le B$. This yields

$$\|w\|_{\infty,E} \le \epsilon + \frac{\delta a^2 b^2}{2(a^2 + b^2)}.$$

The zero-boundary quadratic term attains equality in the constant, so the bound is sharp for this class. Setting $a = b = R$ recovers $\delta R^2/4$. As $b/a \to 0$ with a fixed, the factor behaves as $b^2/2$, displaying how a narrow transverse scale improves a uniform residual bound. This is an exact domain-dependent estimate, not evidence from a finite mesh.

Chapter 3

Poisson kernels and quantitative boundary recovery

3.1 The kernel as a holomorphic real part

Write $\mathbb{T} = \{\zeta : |\zeta| = 1\}$ and normalize angular measure by $dm(\theta) = d\theta/(2\pi)$. For $z \in \mathbb{D}$ and $\zeta = e^{i\theta}$ define

$$P(z,\zeta) = \frac{1-|z|^2}{|\zeta - z|^2} = \operatorname{Re}\frac{\zeta + z}{\zeta - z}.$$

The equality follows after multiplying numerator and denominator by $\overline{\zeta} - \overline{z}$; the real part of the numerator is $1 - |z|^2$. The denominator is nonzero in the open disk, so $P(\cdot,\zeta)$ is harmonic. Also $P > 0$, and the geometric series gives

$$\frac{\zeta + z}{\zeta - z} = 1 + 2\sum_{n=1}^{\infty} (z\overline{\zeta})^n,$$

$$P(re^{it}, e^{i\theta}) = 1 + 2\sum_{n=1}^{\infty} r^n \cos n(t-\theta).$$

For each $r_0 < 1$ the series converges absolutely and uniformly for $r \le r_0$. Integrating term by term proves

$$\int_{\mathbb{T}} P(z,\zeta)\, dm(\zeta) = 1.$$

Thus $P(z,\zeta)\, dm(\zeta)$ is a probability density. If ordinary $d\theta$ is used instead, the factor $1/(2\pi)$ must appear in the density. In particular, $P\, d\theta$ has mass 2π, not one. The distinction is needed in the harmonic-measure and flux formulas later in the book.

3.2 Representation by a change of boundary coordinate

Theorem 3.1 (Poisson representation)**.** *If $u \in \mathcal{H}(\mathbb{D}) \cap C(\overline{\mathbb{D}})$, then*

$$u(z) = \int_{\mathbb{T}} u(\zeta) P(z,\zeta)\, dm(\zeta).$$

For $u \in \mathcal{H}(D(a,R))$ without assumed boundary continuity, the corresponding representation holds on every smaller circle of radius $0 < \rho < R$, with kernel $P((z-a)/\rho, e^{i\theta})$ and boundary values $u(a + \rho e^{i\theta})$, for $|z - a| < \rho$.

Proof. For fixed $z \in \mathbb{D}$, the disk automorphism

$$\phi_z(w) = \frac{w - z}{1 - \overline{z}w}, \qquad \phi_z^{-1}(w) = \frac{z + w}{1 + \overline{z}w}$$

sends z to 0. By conformal covariance, $u \circ \phi_z^{-1}$ is harmonic. The center mean therefore gives $u(z) = \int_{\mathbb{T}} u(\phi_z^{-1}(\eta))\, dm(\eta)$. To change variables, differentiate on the circle:

$$\phi_z'(\zeta) = \frac{1 - |z|^2}{(1 - \overline{z}\zeta)^2}, \qquad |\phi_z'(\zeta)| = \frac{1 - |z|^2}{|\zeta - z|^2} = P(z, \zeta).$$

Since the automorphism preserves orientation, the angular derivative is positive, not merely equal to this quantity in absolute value. One can check the sign without topology: if $\phi_z(e^{i\theta}) = e^{i\psi(\theta)}$, then

$$\psi'(\theta) = \frac{\zeta \phi_z'(\zeta)}{\phi_z(\zeta)} = \frac{1 - |z|^2}{|\zeta - z|^2} > 0.$$

Consequently $d\psi = P(z, \zeta) d\theta$, proving the formula. Translation and dilation reduce a smaller circle to the unit circle. A harmonic function on the original open disk is smooth near every such smaller closed disk, so the boundary-continuity assumption needed for that application is automatic.

One can make the angular change of variables entirely explicit. Write $\phi_z(e^{i\theta}) = e^{i\sigma(\theta)}$ along a continuous lift of its argument. Differentiating gives

$$\sigma'(\theta) = \operatorname{Re} \frac{e^{i\theta} \phi_z'(e^{i\theta})}{\phi_z(e^{i\theta})}.$$

For $|\zeta| = 1$ the identity $(1 - \overline{z}\zeta)(\zeta - z) = \zeta |\zeta - z|^2$ makes the quotient inside the real part equal to $(1 - |z|^2)/|\zeta - z|^2$. Hence $\sigma' = P(z, \zeta) > 0$. A full positive turn in θ is a full positive turn in σ, and $d\sigma = P\, d\theta$. The normalized averaging measure is therefore transported with exactly the stated factor and orientation. □

The smaller-circle clause is essential. A positive harmonic function can blow up at a boundary point and have no continuous trace on $\mathbb{T}$. Applying the unit-boundary formula to such a function without justification would make the subsequent Harnack proof circular or false at its starting point.

Theorem 3.2 (Continuous Dirichlet problem on a disk)**.** *For every real continuous g on $\mathbb{T}$, the function*

$$\mathcal{P}g(z) = \int_{\mathbb{T}} g(\zeta) P(z, \zeta)\, dm(\zeta)$$

is harmonic on $\mathbb{D}$, extends continuously to g on the closed disk, and is the unique such extension. Moreover

$$\min_{\mathbb{T}} g \le \mathcal{P}g \le \max_{\mathbb{T}} g, \qquad \|\mathcal{P}g - \mathcal{P}k\|_{\infty, \mathbb{D}} \le \|g - k\|_{\infty, \mathbb{T}}.$$

Proof. On any compact subset of $\mathbb{D}$, every derivative in z of the kernel is uniformly bounded in ζ. Differentiation under the integral is therefore valid, and $\Delta \mathcal{P}g = 0$. Positivity and unit mass imply both the range bound and the contraction estimate.

Fix a boundary point $\xi \in \mathbb{T}$ and $\varepsilon > 0$. Uniform continuity supplies $\delta > 0$ so that $|g(\zeta) - g(\xi)| < \varepsilon$ when $|\zeta - \xi| < \delta$. If $|z - \xi| < \delta/2$, then on the complementary arc $|\zeta - z| \ge \delta/2$, and hence

$$\int_{|\zeta - \xi| \ge \delta} P(z, \zeta)\, dm(\zeta) \le \frac{4(1 - |z|^2)}{\delta^2}.$$

Splitting the defining integral into the near and far arcs gives

$$|\mathcal{P}g(z) - g(\xi)| \le \varepsilon + \frac{8\|g\|_\infty(1-|z|^2)}{\delta^2}.$$

The second term tends to zero as $z \to \xi$ from any direction in the disk. This proves boundary continuity, with constants uniform in ξ. Uniqueness is the boundary maximum principle applied to the difference of two extensions. □

The existence proof and the representation proof provide two complementary routes. The automorphism argument obtains representation from a known harmonic function and identifies the angular Jacobian. The positive-kernel argument constructs a harmonic function from prescribed continuous data and controls its boundary limit. Neither proof assumes the conclusion of the other: kernel harmonicity and mass were checked directly before both arguments.

3.3 A modulus-of-continuity estimate

Define the chordal modulus

$$\omega_g(\delta) = \sup\{|g(\zeta) - g(\xi)| : \zeta, \xi \in \mathbb{T},\ |\zeta - \xi| \le \delta\}.$$

The previous proof yields, for $|z - \xi| < \delta/2$,

$$|\mathcal{P}g(z) - g(\xi)| \le \omega_g(\delta) + \frac{8\|g\|_\infty(1-|z|^2)}{\delta^2}.$$

This elementary estimate is uniform and effective once a modulus is known, but it is not the sharp radial estimate. To see the boundary scale more accurately, take $z = r\xi$, put $t = 1 - r$, and rotate so that $\xi = 1$. For $r \ge 1/2$ and $-\pi \le \theta \le \pi$,

$$|e^{i\theta} - r|^2 = t^2 + 2r(1 - \cos\theta) \ge t^2 + \frac{2\theta^2}{\pi^2}, \qquad 1 - r^2 \le 2t.$$

Here $1 - \cos\theta = 2\sin^2(\theta/2) \ge 2\theta^2/\pi^2$. If $|g(e^{i\theta}) - g(1)| \le L|\theta|^\alpha$, $0 < \alpha \le 1$, then

$$\begin{aligned} |\mathcal{P}g(r) - g(1)| &\le \frac{2Lt}{\pi}\int_0^\pi \frac{\theta^\alpha}{t^2 + 2\theta^2/\pi^2}\,d\theta \\ &= \frac{2L}{\pi}t^\alpha \int_0^{\pi/t} \frac{s^\alpha}{1 + 2s^2/\pi^2}\,ds. \end{aligned}$$

For $0 < \alpha < 1$, split the last integral at 1 and use $s^\alpha/(1 + 2s^2/\pi^2) \le (\pi^2/2)s^{\alpha-2}$ for $s \ge 1$. Thus a valid explicit bound is

$$|\mathcal{P}g(r) - g(1)| \le C_\alpha L(1-r)^\alpha, \qquad C_\alpha = \frac{2}{\pi}\left(\frac{1}{\alpha+1} + \frac{\pi^2}{2(1-\alpha)}\right).$$

At $\alpha = 1$, the same calculation gives a logarithm rather than a finite integral to infinity:

$$|\mathcal{P}g(r) - g(1)| \le \frac{\pi L}{2}(1-r)\log\left(1 + \frac{2}{(1-r)^2}\right).$$

These constants are useful bounds, not assertions of optimality. The endpoint loss reflects that general Lipschitz data may have a corner at the observed boundary point. More cancellation or differentiability can improve the rate.

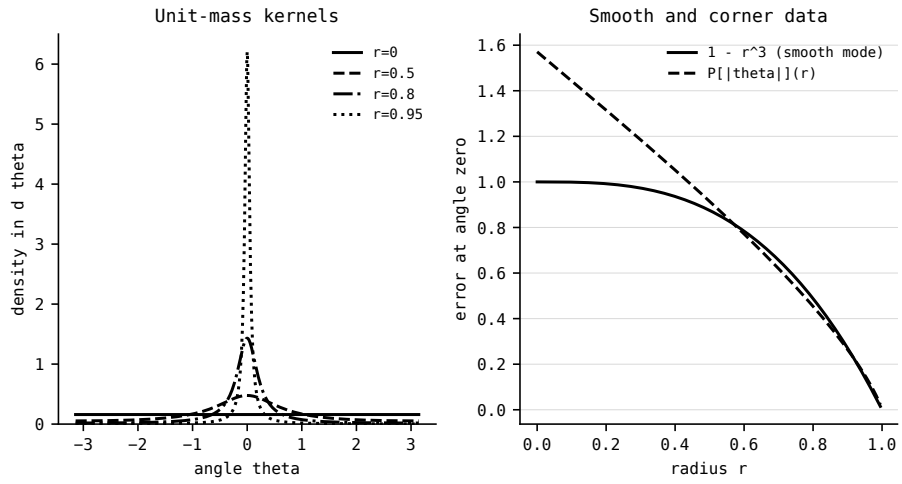


Figure 3.1: The normalized Poisson densities concentrate near the boundary point approached by $r = 0, 0.5, 0.8, 0.95$. The second panel compares radial recovery of a smooth Fourier mode with corner data. The densities have unit mass with respect to $d\theta$ after division by 2π; displayed samples illustrate the kernel and the proved boundary estimates. The corner curve uses the odd Fourier modes through 401 of $|\theta|$; its omitted tail is at most $4r^{403}/(\pi 403^2(1-r^2))$ for $r < 1$.

3.3.1 The boundary scale of a Poisson spike

The concentration estimate becomes more transparent after rescaling the angular variable by the distance to the boundary. Put $r = 1 - t$, $0 < t < 1$, and define, on $|s| < \pi/t$,

$$K_t(s) = \frac{t}{2\pi} P(1-t, e^{its}) = \frac{2-t}{2\pi\,[1 + 2(1-t)(1-\cos(ts))/t^2]}.$$

For each fixed $L < \infty$, Taylor's formula uniformly on $|s| \le L$ gives

$$\frac{2(1-\cos(ts))}{t^2} = s^2 + O(t^2 L^4), \qquad K_t(s) \longrightarrow \frac{1}{\pi(1+s^2)}.$$

This convergence is uniform on that finite interval. To see the denominator control explicitly, use $|2(1-\cos x) - x^2| \le |x|^4/12$. Then its difference from $1 + s^2$ has modulus at most $tL^2 + t^2L^4/12$, while both denominators are at least one. Subtracting the two quotients therefore gives a uniform bound tending to zero without dividing by a vanishing expression.

The rescaled domain grows as $t \downarrow 0$, so local uniform convergence alone does not prove convergence of the total mass. A separate tail estimate supplies that step. For $0 < t \le 1/2$ and $|ts| \le \pi$, $1 - \cos(ts) \ge 2t^2 s^2/\pi^2$, and therefore

$$0 \le K_t(s) \le \frac{1}{\pi(1 + 2s^2/\pi^2)}.$$

Extend K_t by zero outside $[-\pi/t, \pi/t]$. The displayed majorant is integrable on $\mathbb{R}$, so dominated convergence applies. It also gives an explicit tail control, for $L > 0$,

$$\int_{|s|>L} K_t(s)\, ds \le \frac{\pi}{L}.$$

Indeed the majorant is at most $\pi/(2s^2)$ on each tail. The limiting Cauchy density has mass one, in agreement with the original kernel's normalization. The spike is tall on the original angular scale but retains controlled mass after the correct rescaling.

The earlier two-region boundary bound can be optimized without claiming optimal regularity. Suppose $\omega_g(\delta) \le L\delta^\alpha$, $0 < \alpha \le 1$, and $\|g\|_\infty \le M$ with $L, M > 0$. With $A = 1 - r^2$, the crude bound is

$$E(\delta) = L\delta^\alpha + 8MA\delta^{-2}.$$

Its derivative vanishes at $\delta_* = (16MA/(\alpha L))^{1/(\alpha+2)}$. At that value the second term equals $(\alpha/2)L\delta_*^\alpha$, so

$$E(\delta_*) = (1 + \alpha/2)L^{2/(\alpha+2)} \left(\frac{16MA}{\alpha}\right)^{\alpha/(\alpha+2)}.$$

This choice is admissible in the near/far argument when $1 - r \le \delta_*/2$ and $\delta_* \le 2$, which hold sufficiently near the boundary for fixed positive parameters. Its exponent is weaker than the sharper radial Hölder estimate proved earlier. The difference is explained by the coarse treatment of the entire far region: optimizing a valid estimate does not make that estimate sharp.

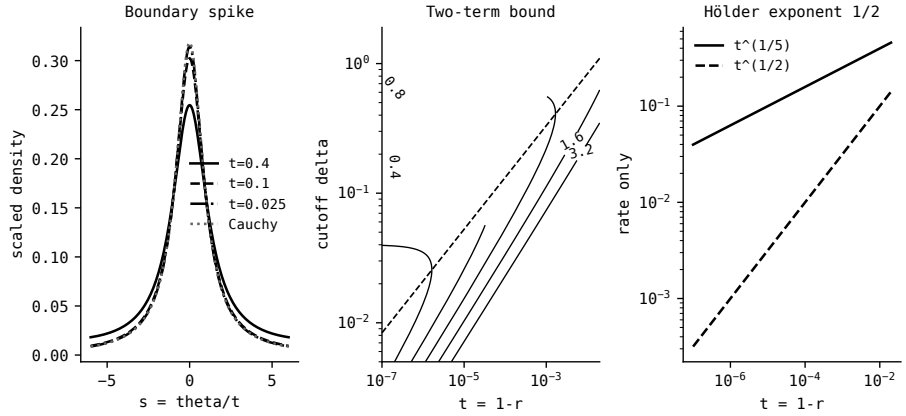


Figure 3.2: Rescaling the Poisson spike by $t = 1 - r$ reveals the limiting density $1/(\pi(1 + s^2))$. The second panel displays the explicit near/far upper bound as a function of boundary distance and angular cutoff; the last panel compares its optimized decay exponent with the sharper radial Hölder rate. Each plotted expression is identified separately, so a bound is not confused with a measured boundary error.

3.4 Schwarz integration and Fourier modes

For continuous real g, define

$$F(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} g(\zeta)\, dm(\zeta).$$

Uniform convergence on compact subsets justifies differentiating this integral. Then F is holomorphic, $\operatorname{Re} F = \mathcal{P}g$ and $\operatorname{Im} F(0) = 0$. This constructs the harmonic conjugate inside the disk explicitly. It does not assert that the conjugate extends continuously to the boundary for every continuous g.

Writing $\widehat{g}(n) = \int_0^{2\pi} g(e^{i\theta}) e^{-in\theta}\, dm(\theta)$, the same geometric series gives

$$F(z) = \widehat{g}(0) + 2\sum_{n\ge1} \widehat{g}(n) z^n, \qquad \mathcal{P}g(re^{it}) = \sum_{n\in\mathbb{Z}} \widehat{g}(n) r^{|n|} e^{int}.$$

For each fixed $r < 1$ this last series is absolutely convergent, since $|\widehat{g}(n)| \le \|g\|_\infty$ and $\sum r^{|n|} < \infty$. No assertion about pointwise convergence of the undamped boundary Fourier series is needed. In particular,

$$\mathcal{P}(\cos n\theta) = r^n \cos nt, \qquad \mathcal{P}(\sin n\theta) = r^n \sin nt \quad (n \ge 1).$$

The factors r^n explain both interior smoothing and the instability of trying to reconstruct fine boundary oscillations from interior values. Later algorithmic estimates will specify finite representations and count the cost of using this damping; the formulas alone do not define a finite input model for an arbitrary continuous function.

For radii $0 < r, s < 1$, the damped kernels satisfy the semigroup identity $P_r * P_s = P_{rs}$ on the circle, with convolution taken against normalized measure. One proof multiplies the absolutely summable Fourier coefficients $r^{|n|}s^{|n|} = (rs)^{|n|}$. Another applies uniqueness twice: first extend the boundary data to radius s, then solve the disk problem at relative radius r; the resulting function agrees with the original solution at radius rs. This identifies the composition without a formal interchange of undamped series at the boundary.

3.5 Worked investigations

Investigation 3.3 (An exact boundary mode)**.** Solve the unit-disk Dirichlet problem for $g(e^{it}) = 3 + 2\cos 2t - \sin 3t$ and find a conjugate normalized to zero at 0.

Solution. The solution and conjugate are

$$u(re^{it}) = 3 + 2r^2\cos 2t - r^3\sin 3t, \qquad v(re^{it}) = 2r^2\sin 2t + r^3\cos 3t.$$

Indeed $u + iv = 3 + 2z^2 + iz^3$. This polynomial is holomorphic on the entire plane, has the prescribed real boundary values, and satisfies $v(0) = 0$. Boundary uniqueness proves that it is the required real solution.

Investigation 3.4 (Mass of a symmetric arc)**.** For $0 \le r < 1$ and $0 < \alpha < \pi$, compute the Poisson mass of $\{e^{it} : |t| < \alpha\}$ seen from r.

Solution. The substitution $s = \tan(t/2)$ gives

$$\omega_r((-\alpha, \alpha)) = \frac{2}{\pi}\arctan\left(\frac{1+r}{1-r}\tan\frac{\alpha}{2}\right).$$

To verify it directly, write the denominator as $(1-r)^2 + 4r\sin^2(t/2)$, use $dt = 2ds/(1+s^2)$, and integrate $2(1-r^2)/((1-r)^2 + (1+r)^2s^2)$ before dividing by 2π. At $r = 0$ this is α/π; as $r \uparrow 1$ it tends to one. This is a formula for continuous-density mass of an arc, not a continuity claim for discontinuous boundary data at the two endpoints.

Investigation 3.5 (The hyperbolic-isometry calculation)**.** Verify the source Exercise 10.6 using $\Psi_\zeta(z) = i(\zeta + z)/(\zeta - z)$, $|\zeta| = 1$.

Solution. Direct calculation gives $\operatorname{Im}\Psi_\zeta(z) = P(z, \zeta)$ and $|\Psi'_\zeta(z)| = 2/|\zeta - z|^2$. With curvature -1 metrics $2|dz|/(1-|z|^2)$ on $\mathbb{D}$ and $|dw|/\operatorname{Im} w$ on $\mathbb{H}$,

$$\frac{|\Psi'_\zeta(z)|}{\operatorname{Im}\Psi_\zeta(z)} = \frac{2}{1-|z|^2}.$$

Thus the map is a hyperbolic isometry and its height function is exactly the Poisson kernel. Harmonicity also follows directly from the imaginary part of this holomorphic map; hyperbolic terminology is not needed for that conclusion.

Investigation 3.6 (Boundary error cannot be improved uniformly)**.** Show that the operator bound $\|\mathcal{P}g - \mathcal{P}k\|_\infty \le \|g - k\|_\infty$ has optimal constant one, even on a fixed smaller disk.

Solution. Choose $g - k = c$ constant and nonzero. Unit mass gives $\mathcal{P}g - \mathcal{P}k = c$ at every interior point, so equality holds. Smoothing damps oscillatory components but leaves the constant mode unchanged. Any improved estimate must remove or separately control that mode.

Investigation 3.7 (A sufficient approach scale)**.** Suppose $\omega_g(\delta) \le L\delta^\alpha$, $0 < \alpha \le 1$. Use the elementary near/far estimate to choose δ in terms of $t = 1 - |z|^2$ and obtain a rate when $|z - \xi| < \delta/2$.

Solution. Balance $L\delta^\alpha$ with $8\|g\|_\infty t/\delta^2$. For $L, \|g\|_\infty > 0$ choose $\delta = (8\|g\|_\infty t/L)^{1/(\alpha+2)}$, provided it is at most two and the stated approach condition holds. Then the error is bounded by $2L^{2/(\alpha+2)}(8\|g\|_\infty t)^{\alpha/(\alpha+2)}$. If $L = 0$, the data are constant and the error is zero. This general near/far estimate is weaker than the sharper radial estimate; the two statements have different geometric information in their hypotheses.

Investigation 3.8 (Damping and inverse amplification)**.** At radius $r_0 \in (0, 1)$, compare the boundary function $\cos n\theta$ with its harmonic trace. What amplification is required to recover that single mode exactly from the interior trace?

Solution. The interior amplitude is r_0^n, whereas the boundary amplitude is one. The inverse map on that one-dimensional mode multiplies by r_0^{-n}. With fixed interior error ε, its reconstructed boundary error can reach εr_0^{-n}. Therefore no uniform bounded inverse exists on all Fourier modes under the sup norm. A finite band limit or an independent regularity condition is essential.

Chapter 4

Harnack geometry, derivative bounds and compactness

4.1 Sharp disk comparison without a boundary trace

Theorem 4.1 (Harnack inequality in the disk)**.** *If $u > 0$ is harmonic on $\mathbb{D}$, then*

$$\frac{1-|z|}{1+|z|}u(0) \le u(z) \le \frac{1+|z|}{1-|z|}u(0).$$

The constants are sharp. The same conclusions hold for nonnegative harmonic functions; such a function is either identically zero or positive everywhere.

Proof. Fix $|z| < \rho < 1$. On the circle of radius ρ, the smaller-circle Poisson formula applies and

$$\frac{\rho-|z|}{\rho+|z|} \le P(z/\rho, e^{it}) \le \frac{\rho+|z|}{\rho-|z|}.$$

Multiplying by the nonnegative values $u(\rho e^{it})$ and integrating gives the same bounds times their average, which is $u(0)$. Let $\rho \uparrow 1$. This argument never uses a boundary value at radius one. For $z = r\xi$, the positive harmonic kernel $P(\cdot, \xi)$ has $P(0, \xi) = 1$ and $P(r\xi, \xi) = (1+r)/(1-r)$, attaining the upper constant. The kernel with pole $-\xi$ attains the lower constant. If a nonnegative harmonic function vanishes at an interior point, the minimum principle makes it zero throughout the connected domain. □

For arbitrary points $z, w \in \mathbb{D}$, set

$$\rho(z,w) = \left|\frac{z-w}{1-\overline{w}z}\right|, \qquad d_{\mathbb{D}}(z,w) = \log\frac{1+\rho(z,w)}{1-\rho(z,w)}.$$

Applying the centered theorem after a disk automorphism gives

$$e^{-d_{\mathbb{D}}(z,w)} \le \frac{u(z)}{u(w)} \le e^{d_{\mathbb{D}}(z,w)}, \qquad |\log u(z) - \log u(w)| \le d_{\mathbb{D}}(z,w).$$

This is an exact statement in the curvature -1 disk metric. Changing the normalization of that metric changes the apparent Lipschitz constant. The bound is not a Euclidean global Lipschitz estimate near the boundary.

Proposition 4.2 (A sharp logarithmic gradient bound)**.** *For a positive harmonic function on $\mathbb{D}$,*

$$|\nabla \log u(z)| \le \frac{2}{1-|z|^2}.$$

Integrating this inequality along a piecewise smooth path recovers the two-point Harnack bound after minimizing hyperbolic length.

Proof. At the center, differentiation of the smaller-circle Poisson formula gives

$$\nabla u(0) = \frac{2}{\rho}\int_0^{2\pi} u(\rho e^{it})(\cos t, \sin t)\, dm(t).$$

For every unit vector e, positivity yields $|e \cdot \nabla u(0)| \le 2u(0)/\rho$. Taking the supremum over e and letting $\rho \uparrow 1$ proves $|\nabla u(0)| \le 2u(0)$. Move a general point to the center. The inverse automorphism has derivative of modulus $1 - |z|^2$ at zero, so the chain rule gives the stated bound. Finally, along a path γ,

$$|\log u(\gamma(1)) - \log u(\gamma(0))| \le \int_\gamma \frac{2|dz|}{1 - |z|^2}.$$

The infimum of these lengths is $d_{\mathbb{D}}$, as established by the disk metric in the preceding hyperbolic volume. The kernel $P(z, 1)$ attains equality in the center gradient bound, so its constant cannot be reduced. □

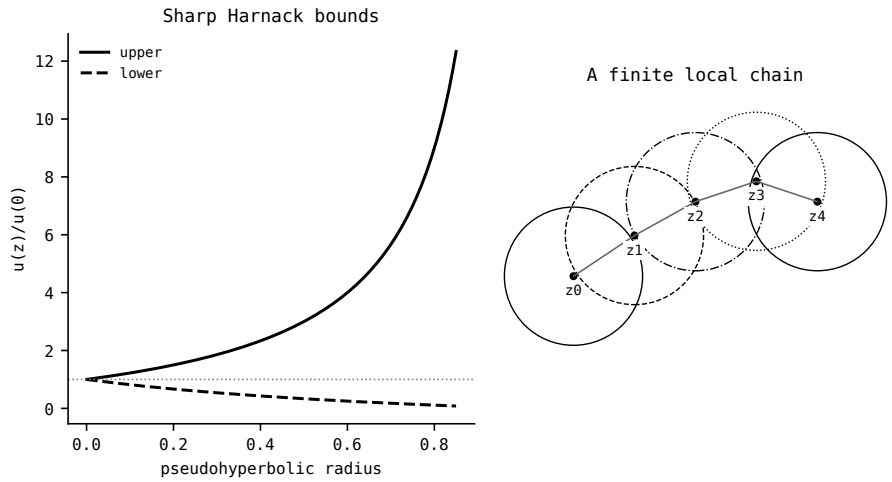


Figure 4.1: Sharp Harnack envelopes $(1 + r)/(1 - r)$ and $(1 - r)/(1 + r)$ surround positive normalized harmonic values. The second panel shows an overlapping chain of coordinate disks; each comparison is local, while the finite product transmits a bound to a compact set.

4.2 Finite chains on an arbitrary connected surface domain

Theorem 4.3 (Compact-set Harnack comparison)**.** *Let U be a connected domain on a Riemann surface, $z_0 \in U$, and $K \Subset U$. There is a finite constant $C = C(U, z_0, K) \ge 1$ such that*

$$C^{-1}u(z_0) \le u(z) \le Cu(z_0) \qquad (z \in K)$$

for every positive harmonic u on U.

Proof. An open connected surface is path connected: the set of points reachable from z_0 by a path is nonempty, open, and has open complement, because coordinate disks are path connected. For each point of K, choose a path from z_0 to that point and cover the path by finitely many coordinate disks compactly contained in U. Compactness of the parameter interval allows the disks to be ordered into a finite overlapping chain, with successive comparison points in smaller concentric disks.

In a chart disk of coordinate radius R, any two points in the concentric radius-$R/2$ disk are each comparable to the center with factor three. Their values are therefore comparable with factor nine. By refining the path cover, all successive points can be chosen in such half disks. A chain of m comparisons gives a factor 9^m. The final half disk provides the same bound in a neighborhood of the terminal point, with at most one extra factor. These neighborhoods cover K; choose a finite subcover and take the maximum of their finitely many comparison constants. That constant depends on the geometry and the chosen compact set, not on u. □

Connectedness of the ambient surface alone is insufficient if the set U on which the function is defined is disconnected. On one component a positive harmonic function can be scaled independently of its value at a basepoint in another component. The source theorem is read with U itself connected. The finite-chain argument makes the required quantifier explicit:

$$(\forall K \Subset U)(\exists C_K < \infty)(\forall u > 0 \text{ harmonic})(\forall z \in K) \quad C_K^{-1} u(z_0) \le u(z) \le C_K u(z_0).$$

There is no assertion that one constant works for all compact subsets approaching an arbitrary boundary.

4.3 Interior derivatives and normal families

Lemma 4.4 (An interior gradient estimate)**.** *If u is harmonic on a neighborhood of $\overline{D(a,R)}$ and $|u| \le M$ there, then $|\nabla u(a)| \le 2M/R$. On a compact subset of an open domain, a locally uniform bound on $|u|$ yields uniform bounds on derivatives of every fixed order on smaller compact subsets.*

Proof. The differentiated Poisson formula used above remains valid without positivity. Project its gradient on any unit vector and use the bound $|u| \le M$ to obtain $2M/R$. For higher derivatives, take an interior disk of radius R and differentiate the kernel at points in the disk of radius $R/2$. The denominator is bounded below by a positive multiple of R, and each differentiated rational expression has a finite bound depending only on its order and R. Equivalently, expand the kernel in its absolutely convergent power series and differentiate on the smaller disk; the sums $\sum_{n \ge k} n^k 2^{-n}$ converge. A finite disk cover then gives the corresponding compact-set bounds. □

Theorem 4.5 (Normality of locally bounded harmonic families)**.** *Let $\mathcal{F} \subset \mathcal{H}(U)$ be locally uniformly bounded in absolute value: for every $K \Subset U$ there is $M_K < \infty$ with $|u| \le M_K$ on K for every $u \in \mathcal{F}$. Every sequence in $\mathcal{F}$ has a subsequence converging uniformly on compact subsets to a harmonic function. The derivatives of each fixed order converge uniformly on smaller compact subsets along that subsequence.*

Proof. Choose a countable compact exhaustion of U by sets contained in finitely many coordinate disks; such an exhaustion exists for the second-countable surfaces considered here. Enlarge each compact set slightly inside U. The local bound on the enlargement and the gradient estimate give uniform boundedness and equicontinuity on the original compact set. The Arzelà–Ascoli theorem supplies a uniformly convergent subsequence there. Choose successively nested subsequences for the exhaustion and take the diagonal subsequence. For each fixed exhaustion set, the tail belongs to its selected subsequence and converges uniformly. The limits agree on overlaps and define a continuous function u on U.

On any closed coordinate disk inside U, the mean-value identity passes to the limit. Lemma 2.2 proves that u is harmonic. For derivative convergence, choose a slightly larger concentric disk. Express each derivative on the smaller disk by the derivative of the Poisson kernel integrated against boundary values on the larger circle. The kernel derivative is uniformly bounded there and the boundary values converge uniformly. Subtracting the two integrals proves uniform convergence of the derivatives, rather than merely their boundedness. □

Normality here is a statement about subsequences and a specified topology. It does not imply convergence of the original sequence. The functions $u_n = (-1)^n$ are uniformly bounded and harmonic but do not converge. It also does not follow from a bound at one point for arbitrary signed harmonic functions: $u_n(z) = n \operatorname{Re} z$ all vanish at zero and are unbounded on every disk. Positivity is what converts one-point control to local absolute bounds through Harnack's inequality.

4.3.1 Derivative constants and the diagonal selection

Throughout this subsection, u is harmonic on $D(a, R)$ and $|u| \le M$ there. The compactness argument can be expressed using explicit local constants. If $|u| \le M$ on $D(a, R)$, take a circle of radius $\rho < R$ and write its real Fourier series in the form

$$u(a + z) = u(a) + \operatorname{Re} \sum_{n \ge 1} A_n z^n, \qquad A_n = \frac{1}{\pi \rho^n} \int_0^{2\pi} u(a + \rho e^{i\theta}) e^{-in\theta} \, d\theta.$$

The integral representation gives $|A_n| \le 2M/\rho^n$. For $p + q = n \ge 1$,

$$\partial_x^p \partial_y^q u(a) = \operatorname{Re}(i^q n! A_n), \qquad |\partial_x^p \partial_y^q u(a)| \le \frac{2n!M}{\rho^n}.$$

Letting $\rho \uparrow R$ is allowed because the left side does not depend on ρ. Applying the same center estimate in a disk around z gives

$$|D^\beta u(z)| \le \frac{2|\beta|!M}{(R - |z - a|)^{|\beta|}}, \qquad |\beta| \ge 1.$$

No bound on an arbitrarily normalized harmonic conjugate was used. Bounding only the real part of its holomorphic primitive is sufficient because its nonconstant coefficients have the displayed Fourier representation.

For the diagonal extraction on a general planar domain, one possible compact exhaustion is

$$K_m = \{z \in U : |z| \le m, \ \operatorname{dist}(z, \mathbb{C} \setminus U) \ge 1/m\}.$$

For $U = \mathbb{C}$, the distance to the empty complement is interpreted as infinity. Each compact subset of U lies in some K_m. Choose a slightly larger compact neighborhood of each nonempty K_m and finitely many disks within it. Local uniform bounds there and the first-derivative estimate imply uniform equicontinuity on K_m. Arzelà–Ascoli selects a subsequence on K_1, then a subsequence of that subsequence on K_2, and so on. If $n_j^{(m)}$ denotes the mth nested subsequence, choose the diagonal indices recursively so they are increasing and its jth term belongs to the jth subsequence. For fixed m, all sufficiently late diagonal terms belong to the mth subsequence and hence converge uniformly on K_m.

The local limits agree on overlaps because they are limits of the same pointwise diagonal sequence. They therefore define one continuous function on U. Its circle means pass to the limit on each fixed interior circle, so it is harmonic. For derivative convergence, apply the displayed bound to the harmonic differences $u_{n_j} - u$ on a slightly larger disk. The supremum of those differences tends to zero there, and thus every fixed order derivative tends to zero uniformly on a smaller disk. This proves all derivative conclusions from a single compact-uniform subsequence; separate unrelated subsequences for different derivatives are unnecessary.

4.4 Monotone convergence and an explicit Cauchy estimate

Theorem 4.6 (Harnack convergence principle)**.** *Let u_n be an increasing sequence of harmonic functions on a connected domain U. Exactly one of the following occurs:*

(i) $u_n \to u$ *uniformly on compact subsets, with* u *finite and harmonic;*
(ii) $u_n(z) \to +\infty$ *for every* $z \in U$*, uniformly from below on each compact subset.*
If $u_n(z_0)$ *is bounded above at one point, case (i) holds.*

Proof. Set $v_n = u_n - u_1 \geq 0$. If $v_n(z_0)$ is bounded, compact-set Harnack comparison bounds v_n on every compact set. More directly, for $n \geq m$ the nonnegative harmonic difference satisfies

$$0 \leq u_n(z) - u_m(z) \leq C_K\big(u_n(z_0) - u_m(z_0)\big) \qquad (z \in K).$$

The scalar increasing bounded sequence at z_0 is Cauchy. Thus the displayed estimate makes u_n uniformly Cauchy on each compact set. Its compact-uniform limit is harmonic by the mean-value argument. Letting $n \to \infty$ also gives the useful bound

$$0 \leq u(z) - u_m(z) \leq C_K\big(u(z_0) - u_m(z_0)\big).$$

If instead $v_n(z_0) \to \infty$, the lower Harnack bound gives $v_n(z) \geq C_K^{-1} v_n(z_0)$ on K. Adding the fixed continuous u_1 proves divergence uniformly from below. If the finite alternative holds at any other point, the two-point comparison forces boundedness at z_0 as well, so these cases exhaust all possibilities. □

The proof gives a rate only if the one-point tail is itself known. A theorem that an increasing sequence converges does not automatically provide an effective stopping rule for computing its limit. This distinction will reappear in the Perron supremum: harmonicity can be proved even when no finite description of all admissible subfunctions is available.

4.5 Worked investigations

Investigation 4.7 (A sharp ratio)**.** For a positive harmonic function on $\mathbb{D}$ with $u(0) = 2$, bound $u(1/3)$ and produce examples attaining both endpoints.

Solution. The Harnack factors are 2 and $1/2$, so $1 \leq u(1/3) \leq 4$. The functions $2P(z, 1)$ and $2P(z, -1)$ attain 4 and 1 respectively. They are positive and harmonic in the open disk although unbounded at one boundary point; this is why a continuous boundary trace cannot be required in the sharp general inequality.

Investigation 4.8 (Three comparisons)**.** Suppose a path can be traversed using three half-disk comparisons of the type used above. Give a valid ratio bound and explain why it need not be sharp.

Solution. Each comparison has ratio at most nine in either direction, so $9^{-3} \leq u(z)/u(z_0) \leq 9^3$. The separate disk estimates may have incompatible equality cases, and the path cover may be inefficient. The product is a valid geometric bound, not an extremal value for the domain.

Investigation 4.9 (A one-point bound without positivity)**.** Construct harmonic functions bounded at a fixed point but with no compact-uniformly convergent subsequence on a neighborhood of that point.

Solution. Take $u_n(z) = n \operatorname{Re} z$. At zero all values vanish. At a fixed positive real r, however, $u_n(r) = nr \to \infty$ along every subsequence. Uniform convergence on a disk containing r is impossible. The failure is not a failure of normality under its stated hypothesis: the family is not locally uniformly bounded in absolute value.

Investigation 4.10 (Transferring a monotone rate)**.** Suppose $u_n \uparrow u$, $0 \leq u(z_0) - u_n(z_0) \leq 2^{-n}$, and a compact-set Harnack constant is $C_K = 27$. How many terms suffice for $\|u - u_n\|_{\infty,K} \leq 10^{-3}$?

Solution. It suffices that $27 \cdot 2^{-n} \le 10^{-3}$, or $2^n \ge 27000$. Since $2^{14} = 16384$ and $2^{15} = 32768$, $n = 15$ suffices. This is a proved stopping rule because the one-point rate and the geometric comparison constant were supplied, not merely observed.

Investigation 4.11 (A logarithmic gradient extremizer)**.** For $u(z) = P(z, 1)$, calculate $\nabla \log u(0)$ and compare its norm with the general bound.

Solution. Since $\log u = \log(1 - x^2 - y^2) - \log((1 - x)^2 + y^2)$, its gradient at zero is $(2, 0)$. Its norm is two, equal to $2/(1 - |0|^2)$. Along the real diameter, $u(r) = (1 + r)/(1 - r)$ and $d(\log u)/dr = 2/(1 - r^2)$, so equality persists along that geodesic direction.

Investigation 4.12 (Why local uniformity matters)**.** Consider $u_n(z) = \operatorname{Re} z^n$ on the disk. Determine its compact limit and explain why convergence is not uniform on the open disk.

Solution. On $|z| \le r < 1$, $|u_n(z)| \le r^n \to 0$, so the compact limit is zero and is harmonic. Yet $\sup_{|z|<1} |u_n(z)| = 1$ for every n, approached along the positive real radius. Compact convergence leaves a boundary layer uncontrolled, and cannot be replaced by convergence in the supremum norm on the entire open disk.

Chapter 5

Isolated singularities and spectral information

5.1 A puncture is not automatically a boundary component

The circle averages of a harmonic function on an annulus encode a topological obstruction that no local conjugate can remove. Suppose u is harmonic on $0 < |z| < R$, and define

$$m(r) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})\, d\theta.$$

Integrating the polar Laplace equation, justified on each compact annulus, gives

$$m''(r) + r^{-1} m'(r) = 0, \qquad (rm'(r))' = 0, \qquad m(r) = A \log r + B.$$

Thus the coefficient A is not an informal growth descriptor: it is the normalized, radius-independent flux

$$A = \frac{1}{2\pi} \int_{|z|=r} d^c u = \frac{1}{2\pi} \int_0^{2\pi} r \partial_r u(re^{i\theta})\, d\theta. \tag{5.1}$$

For a positively oriented circle this is outward flux from its disk; as the inner boundary of an annulus the same geometric circle receives the opposite orientation. Both the orientation and the factor r are essential. A logarithmic singularity can therefore be detected without choosing a branch of a conjugate.

Theorem 5.1 (Bounded harmonic punctures are removable)**.** *Let u be real harmonic on $0 < |z| < R$. If $|u(z)| \le M$ there, then there is a unique harmonic function on $|z| < R$ whose restriction to the punctured disk is u.*

Proof. Choose $0 < \rho < R$ and let H be the Poisson solution on $|z| < \rho$ with boundary values $u|_{|z|=\rho}$. Its absolute value is at most M. The difference $w = u - H$ is harmonic off zero, vanishes on the outer circle, and satisfies $|w| \le 2M$. For $0 < \epsilon < |z| < \rho$, define

$$b_\epsilon(z) = \frac{\log(\rho/|z|)}{\log(\rho/\epsilon)}.$$

This function is harmonic on the annulus and equals one on its inner circle and zero on its outer circle. Boundary comparison therefore gives

$$-2Mb_\epsilon(z) \le w(z) \le 2Mb_\epsilon(z).$$

For fixed $z \neq 0$, the numerator is fixed and the denominator tends to infinity as $\epsilon \downarrow 0$. Hence $w(z) = 0$. Set $u(0) = H(0)$. The resulting extension agrees with H throughout $|z| < \rho$, and is therefore harmonic across zero. Different choices of ρ agree on their punctured overlap and then at zero by continuity. Uniqueness follows in the same way. □

There is a second useful route. On a simply connected slit disk choose a conjugate and write a holomorphic function F with real part u. Then e^F has bounded modulus above and below. The obstruction is its monodromy: boundedness of u first implies $A = 0$ in (5.1), so the conjugate has zero period and F is actually single valued on the punctured disk. The bounded holomorphic function e^F extends across zero by the holomorphic removable-singularity theorem; its extension is nonzero because $|e^F| \geq e^{-M}$. A local holomorphic logarithm supplies the desired harmonic extension of u. This route uses the previously established complex-analytic removable theorem, whereas the logarithmic-barrier proof uses only disk Dirichlet solutions and comparison.

5.2 Positive singularities have only a logarithmic principal part

Theorem 5.2 (Planar positive-puncture decomposition)**.** *If $u \geq 0$ is harmonic on $0 < |z| < R$, then*

$$u(z) = c \log \frac{1}{|z|} + h(z), \qquad c \geq 0,$$

where h is harmonic on the whole disk. The coefficient is $c = -(2\pi)^{-1} \int_{|z|=r} d^c u$.

Proof. We give the Fourier argument, including the step that excludes angular principal parts. For $n \in \mathbb{Z}$ put

$$c_n(r) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) e^{-in\theta} \, d\theta.$$

Positivity implies $|c_n(r)| \leq c_0(r) = A \log r + B$. The right side is nonnegative for arbitrarily small r, so $A \leq 0$. For $n \geq 1$, differentiating under the integral and integrating by parts twice in θ yields the Euler equation

$$c_n'' + r^{-1} c_n' - n^2 r^{-2} c_n = 0, \qquad c_n(r) = a_n r^n + b_n r^{-n}.$$

Multiplying by r^n and letting $r \downarrow 0$, the bound $r^n |A \log r + B| \to 0$ forces $b_n = 0$. Real-valuedness gives $c_{-n} = \overline{c_n}$. Fix $0 < \rho < R$ and let h be the Poisson extension of $u(\rho e^{i\theta}) - A \log \rho$ to that disk. Its zeroth coefficient is B, and its nth coefficient at radius r is $a_n r^n$ for $n > 0$, by the Poisson Fourier multiplier. Thus $u(re^{i\theta}) - A \log r$ and $h(re^{i\theta})$ have identical Fourier coefficients for every $0 < r < \rho$. Their difference is a smooth periodic function, so its uniformly convergent Fourier series is zero. They agree pointwise. The disk extensions for different ρ agree on overlaps. Set $c = -A \geq 0$.

The limit is justified, for example, by setting $r = e^{-s}$: then $r^n |\log r| = s e^{-ns} \to 0$ as $s \to \infty$. The Euler equation itself can be reduced to a constant-coefficient equation by writing $y_n(s) = c_n(e^s)$. The chain rule yields $y_n'' = r^2 c_n'' + r c_n'$, so $y_n'' - n^2 y_n = 0$ and its two solutions are e^{ns} and e^{-ns}. This proves the asserted two-term radial form without assuming a Laurent expansion for a globally single-valued conjugate.

One may justify the uniqueness of angular Fourier data using only the disk Poisson theorem already established. If a continuous periodic difference v has every Fourier coefficient zero, then its Poisson integral at each radius $s < 1$ is zero, by the absolutely convergent kernel series. Boundary convergence of that integral gives $v = 0$ on the circle. This avoids assuming pointwise convergence of a Fourier series for arbitrary continuous data. In the present proof the angular functions are smooth, but this stronger uniqueness argument clarifies exactly which result is needed.

□

The positivity assumption has real content. The function $\mathrm{Re}(1/z) = r^{-1} \cos\theta$ has zero circular mean and zero logarithmic flux, but has an unbounded sign-changing singularity. Zero flux alone removes the logarithmic term, not every singular term. Conversely, a harmonic function bounded

below near a puncture has the same decomposition after adding a constant. A function bounded both above and below must have $c = 0$, recovering removability.

5.3 Angular modes and what an interior measurement loses

For a harmonic function continuous on the closed disk of radius R, the Fourier representation can be written

$$u(re^{i\theta}) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n (a_n \cos n\theta + b_n \sin n\theta), \qquad r < R. \tag{5.2}$$

The boundary coefficients are

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\, d\theta, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos n\theta\, d\theta, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin n\theta\, d\theta.$$

For every fixed $r < R$, absolute and uniform convergence follows from $|a_n|, |b_n| \le 2\|g\|_\infty$ and the geometric series. The same estimate permits differentiation any fixed number of times on a smaller disk, since $\sum n^k s^n < \infty$ for $0 < s < 1$. There is no need to assert absolute convergence of the boundary Fourier series for arbitrary continuous g.

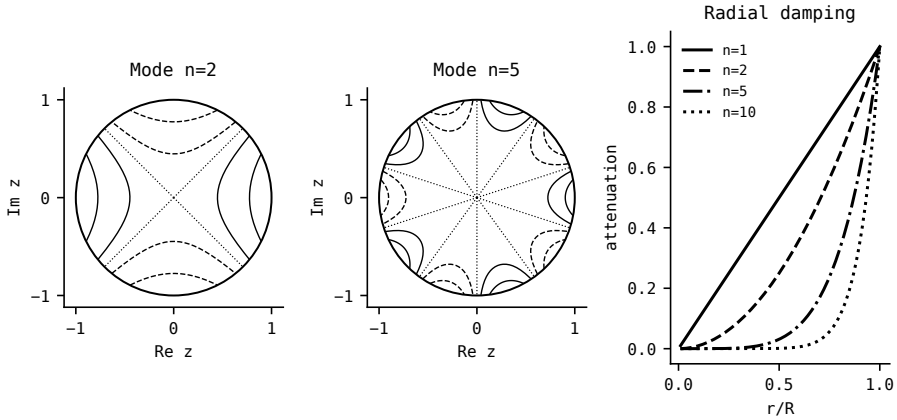


Figure 5.1: Angular harmonic modes and radial attenuation. Nodal rays and level curves are exact level sets of $r^n \cos(n\theta)$; the attenuation panel plots $(r/R)^n$. Higher angular frequencies are suppressed more strongly at interior radii. These are formula-based illustrations, not measured data.

For smooth boundary data the outward normal derivative has Fourier coefficients

$$\partial_n u(Re^{i\theta}) = \frac{1}{R}\sum_{n\ge 1} n(a_n \cos n\theta + b_n \sin n\theta).$$

This is the disk Dirichlet-to-Neumann map: constant boundary data have zero flux, while the nth angular frequency has eigenvalue n/R. For trigonometric polynomials, Green's identity and orthogonality give the exact energy formula

$$\int_{|z|<R} |\nabla u|^2\, dA = \int_{|z|=R} u\partial_n u\, ds = \pi \sum_{n\ge 1} n(a_n^2 + b_n^2). \tag{5.3}$$

For an infinite series this identity holds whenever the weighted sum is finite, interpreted by increasing radii and monotone convergence of the nonnegative energy terms. More explicitly, the

energy on $|z| < sR$ is $\pi \sum n(a_n^2 + b_n^2)s^{2n}$, obtained first on compact subdisks where derivatives converge uniformly. Letting $s \uparrow 1$ proves the assertion, including the possibility of infinite energy. Continuity of the boundary function by itself does not imply finite energy.

The attenuation in (5.2) also quantifies an inverse problem. If a mode is observed on $|z| = r_0 < R$ with coefficient error η_n, reconstruction of its boundary coefficient multiplies that error by $(R/r_0)^n$. Consequently the forward harmonic-extension map is stable in the supremum norm, but unrestricted inversion from a strictly interior circle is unstable. Indeed, the boundary functions $g_n(\theta) = \cos n\theta$ have norm one, while their restrictions after harmonic extension have norm $(r_0/R)^n \to 0$. There cannot be a constant C satisfying

$$\|g\|_\infty \le C\|u|_{|z|=r_0}\|_\infty$$

for all such data. This is a mathematical obstruction, not a failure of a particular reconstruction algorithm. Any regularized inverse needs an additional bandwidth or smoothness assumption.

5.3.1 A regularized inverse with a complete error budget

The instability of unrestricted inversion leaves open a precise, conditional recovery problem. Let the boundary data have complex Fourier coefficients c_n with $c_{-n} = \overline{c_n}$ and suppose a supplied prior gives

$$|c_n| \le Mq^{|n|} \quad (n \ne 0), \qquad M > 0, \quad 0 < q < 1.$$

This exponential prior implies absolute boundary convergence. It is an additional hypothesis, not a consequence of mere continuity. On an interior circle of relative radius $s \in (0,1)$, the exact coefficients are $d_n = s^{|n|}c_n$. Suppose approximate coefficients satisfy $|\widetilde{d}_n - d_n| \le \eta$ for $|n| \le N$ and are conjugate symmetric. The truncated reconstruction is

$$\widetilde{g}_N(\theta) = \sum_{|n|\le N} s^{-|n|}\widetilde{d}_n e^{in\theta}.$$

Subtract the exact boundary series before taking absolute values. The retained coefficients and omitted tail give separate sums:

$$\begin{aligned} |\widetilde{g}_N(\theta) - g(\theta)| &\le \sum_{|n|\le N} s^{-|n|}|\widetilde{d}_n - d_n| + \sum_{|n|>N} |c_n| \\ &\le \eta\left(1 + 2\sum_{n=1}^{N} s^{-n}\right) + \frac{2Mq^{N+1}}{1-q}. \end{aligned}$$

The bound is uniform in θ. There is no random-noise assumption and no cancellation is presumed between coefficient errors. If the interior trace itself is known in supremum norm within η, integration against each unit-modulus Fourier character proves the required coefficient error bound. An actual quadrature or rounding error must be added separately if those coefficients are computed approximately.

The geometric sum is explicit:

$$\sum_{n=1}^{N} s^{-n} = \frac{s^{-N} - 1}{1 - s},$$

with zero sum when $N = 0$. Increasing N decreases the prior tail but increases the possible error amplification. For $0 < \eta \le M$, set

$$T = \frac{\log(M/\eta)}{\log(1/(qs))}, \qquad N = \lfloor T \rfloor, \qquad \alpha = \frac{\log(1/q)}{\log(1/(qs))} \in (0,1).$$

Since $N \le T < N+1$, the two inequalities $s^{-N} \le s^{-T}$ and $q^{N+1} \le q^T$ hold in the useful directions. Direct exponentiation gives

$$\eta s^{-T} = M^{1-\alpha}\eta^{\alpha}, \qquad Mq^T = M^{1-\alpha}\eta^{\alpha}.$$

Consequently the reconstruction has the proved bound

$$\|\widetilde{g}_N - g\|_{\infty} \le \eta + \left(\frac{2}{1-s} + \frac{2}{1-q}\right) M^{1-\alpha}\eta^{\alpha}.$$

The exponent deteriorates as the observation circle moves inward. For fixed s, a stronger decay prior (smaller q) improves it. At $\eta = 0$ one may let $N \to \infty$ and recover exactly; the formula using $\log(M/\eta)$ is not evaluated at that endpoint. Without the prior, the tail term is uncontrolled and the earlier instability example remains decisive.

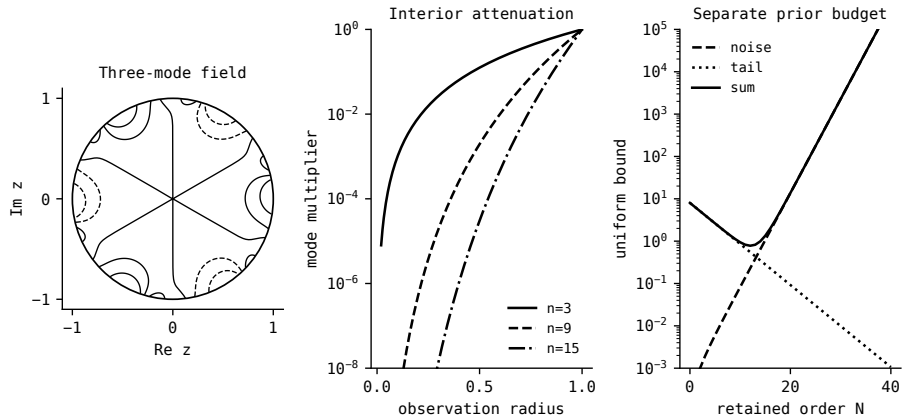


Figure 5.2: A finite harmonic field with boundary data $\cos 3\theta + \frac{1}{2}\cos 9\theta + \frac{1}{4}\sin 15\theta$ shows how angular structure disappears inward. The attenuation panel follows its exact Fourier multipliers. A separate reconstruction-budget panel uses $s = 0.6$, $q = 0.8$, $M = 1$ and a declared coefficient error 10^{-4} to show the competing amplified-error and prior-tail terms. No measurement or executed inverse experiment is represented.

The energy formula also admits a direct calculation that exposes the weights. For one cosine mode $u = a(r/R)^n \cos n\theta$,

$$u_r = \frac{na}{R^n} r^{n-1} \cos n\theta, \qquad r^{-1}u_\theta = -\frac{na}{R^n} r^{n-1} \sin n\theta.$$

Thus $|\nabla u|^2 = n^2 a^2 r^{2n-2}/R^{2n}$, and radial integration gives

$$\int_0^{2\pi}\int_0^R |\nabla u|^2 r\, dr\, d\theta = \frac{2\pi n^2 a^2}{R^{2n}} \frac{R^{2n}}{2n} = \pi n a^2.$$

Sine modes have the same energy. The angular integrals of cross terms between distinct frequencies vanish; the sine/cosine cross term at the same frequency also vanishes after adding its radial and angular contributions. This yields the full sum in (5.3) directly. On a smaller disk $r < sR$, absolute convergence of differentiated series justifies those integrations; increasing s to one then gives the finite- or infinite-energy alternative by monotone convergence. A truncation's energy deficit is exactly $\pi \sum_{n>N} n(a_n^2 + b_n^2)$ when that sum is finite, whereas its supremum error is controlled by a different, unweighted coefficient tail. Neither error measure can be silently substituted for the other.

5.4 Reflection across a straight boundary segment

Suppose u is harmonic in the upper half of a disk, continuous on its closure, and zero along the diameter. One can prove harmonic reflection without assuming differentiability on that diameter. Prescribe continuous data on the full boundary circle by taking the given upper-semicircle values and their negative reflection on the lower semicircle. The values match at the endpoints because the diameter data vanish. Let H be the full-disk Poisson solution. The function $\widetilde{H}(z) = -H(\overline{z})$ has the same circle data as H; uniqueness implies $\widetilde{H} = H$, hence $H = 0$ on the diameter. Upper-half-disk comparison now gives $H = u$ there. Thus

$$U(x,y) = \begin{cases} u(x,y), & y \geq 0, \\ -u(x,-y), & y < 0 \end{cases}$$

is harmonic across the open diameter. Local half-disks prove the same statement along any straight open segment. A conformal coordinate straightening an analytic boundary arc transfers the result to that arc, provided the coordinate extends conformally across it. The latter extension is an additional geometric hypothesis; no claim of analytic reflection across an arbitrary continuous boundary is made.

5.5 Worked investigations

Investigation 5.3 (Flux is not boundedness)**.** Compute the flux of $u(z) = \mathrm{Re}(z^{-2})$ around a puncture. Does zero flux imply a removable singularity?

Solution. Here $u = r^{-2}\cos 2\theta$, so $r\partial_r u = -2r^{-2}\cos 2\theta$ and its angular integral is zero. Along the positive real axis $u = r^{-2} \to +\infty$, while along the positive imaginary axis $u = -r^{-2} \to -\infty$. The singularity is not removable. Its obstruction is an angular principal part rather than a logarithmic one, precisely the possibility excluded by positivity in Theorem 5.2.

Investigation 5.4 (A logarithmic coefficient from two averages)**.** A harmonic function on $1/8 < |z| < 1$ has circular means $m(1/2) = 3$ and $m(1/4) = 5$. Determine its logarithmic coefficient and flux.

Solution. Since $m(r) = A\log r + B$, $2 = A\log(1/2) = -A\log 2$, so $A = -2/\log 2$ and $B = 1$. Its positive-circle flux is $2\pi A = -4\pi/\log 2$. The two averages determine neither the angular Fourier coefficients nor the full function. For example, any added harmonic mode with zero circular mean preserves both observations and the flux.

Investigation 5.5 (Removability with an explicit comparison error)**.** Let $|u| \leq 4$ on the punctured unit disk and let H solve its boundary problem on $|z| = 1/2$. Bound $|u(z) - H(z)|$ at $|z| = 1/4$ using an inner comparison circle of radius 2^{-k}, $k > 2$.

Solution. The barrier proof gives

$$|u(z) - H(z)| \leq 8\frac{\log((1/2)/(1/4))}{\log((1/2)/2^{-k})} = \frac{8}{k-1}.$$

The inequality holds for every integer $k > 2$. Sending k to infinity proves exact equality $u(z) = H(z)$; selecting a single large k would give only a bound. The proof therefore uses an entire family of comparisons, not one finite computation.

Investigation 5.6 (Energy and scale)**.** For $u(r,\theta) = 3(r/R)^2\cos 2\theta - 2(r/R)^5\sin 5\theta$, calculate the disk energy and the boundary normal derivative.

Solution. Orthogonality gives $E = \pi(2 \cdot 9 + 5 \cdot 4) = 38\pi$. At the boundary, $\partial_n u = R^{-1}(6\cos 2\theta - 10\sin 5\theta)$. The derivative scales like inverse length, the boundary element scales like length, and the energy remains independent of R. This directly checks the two-dimensional conformal scale cancellation.

Investigation 5.7 (A quantitative inverse cutoff)**.** Measurements at $r_0 = R/2$ have absolute coefficient errors at most 10^{-6}. Which mode indices can be recovered with a worst-case coefficient error at most 10^{-2}, without using a prior?

Solution. The propagated error is at most $2^n 10^{-6}$, so it is enough that $2^n \le 10^4$. Thus $n \le 13$ satisfies the bound, while $n = 14$ does not. This statement concerns supplied deterministic coefficient errors. It does not assert that actual measurements or a numerical transform attain those errors, and it says nothing about unknown modes above the cutoff without a tail assumption.

Investigation 5.8 (Reflection of a polynomial mode)**.** Reflect $u(x, y) = 2xy$ from the upper half-disk across its zero diameter, and contrast it with $v(x, y) = 1 + y$.

Solution. Odd reflection gives $-u(x, -y) = 2xy$, hence the same harmonic polynomial throughout the disk. The function v has nonzero diameter trace. Its naive odd reflection jumps from 1 to -1 there, so it is not even continuous. One may first subtract its constant trace: $v - 1 = y$ reflects harmonically, and adding the constant back yields $1 + y$. For nonconstant boundary traces, subtracting a harmonic extension of the trace requires a separate construction.

Investigation 5.9 (A positive pole and its regular part)**.** On $0 < |z| < 1/2$, consider $u(z) = 2\log(1/|z|) + \operatorname{Re} z$. Verify positivity, identify the puncture coefficient, and compute its flux.

Solution. Since $\operatorname{Re} z \ge -|z| > -1/2$ and $2\log(1/|z|) > 2\log 2 > 1/2$, the function is positive. Its coefficient is $c = 2$, its regular part is $h(z) = \operatorname{Re} z$, and its flux is $-2\pi c = -4\pi$. The regular part is unique because two decompositions would have a logarithmic difference harmonic and bounded at zero, forcing the logarithmic coefficient to vanish.

Chapter 6

Subharmonicity as a comparison inequality

The mean-value identity imposes a balance at every scale. Replacing equality by one inequality permits sources, corners, and logarithmic singularities while retaining a useful direction of comparison. The direction will remain fixed throughout this volume:

$$\Delta u \geq 0, \qquad u(a) \leq \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{it})\,\mathrm{d}t, \qquad u \leq \text{its harmonic boundary extension.}$$

These are three expressions of the same principle under the hypotheses proved below. None permits an upward isolated spike. Downward infinite values, on the other hand, are necessary: $\log|z|$ is the prototype, with value $-\infty$ at zero. A theorem about such functions must explain both the pointwise value and the meaning of its integrals.

Source correspondence. Lyubich, Section 10.5, Definition 10.11 and Remark 10.12.

We work on a connected open set $\Omega \subset \mathbb{C}$. Circle integrals initially mean extended Lebesgue integrals. An upper bound on a compact circle makes the positive part integrable, so a mean can be finite or $-\infty$, but never an undefined difference $\infty - \infty$. The elementary measure inputs used here are monotone convergence, Tonelli's theorem for nonnegative functions, and Lebesgue integration in polar coordinates. Their hypotheses are identified when singular integrands occur. We write $dA = dx\,dy$ for Euclidean area.

6.1 Upper semicontinuity and the submean condition

Definition 6.1 (Pointwise convention)**.** A function $u : \Omega \to [-\infty, \infty)$ is upper semicontinuous if

$$\limsup_{z \to a} u(z) \leq u(a) \quad (a \in \Omega).$$

It is subharmonic if it is upper semicontinuous, is not identically $-\infty$, and satisfies

$$u(a) \leq M_u(a, r) := \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{it})\,\mathrm{d}t \quad \text{whenever } \overline{D(a, r)} \subset \Omega.$$

A function $v : \Omega \to (-\infty, \infty]$ is superharmonic if $-v$ is subharmonic. On disconnected open sets the nontriviality condition is imposed on each component separately.

At a finite point, upper semicontinuity means that for every $\varepsilon > 0$ there is a neighborhood on which $u < u(a) + \varepsilon$. At a point with value $-\infty$, it means that for every real B there is a neighborhood on which $u < B$. Consequently u has a finite upper bound on every compact

subset: choose an upper bound on a neighborhood of each point and take the maximum of finitely many bounds from a finite subcover. The sets $\{u < c\}$ are open, so u is Borel measurable. These two facts justify the circle integrals in the definition before local integrability has been established.

Lemma 6.2 (Smooth criterion)**.** *For a real C^2 function u on Ω, subharmonicity is equivalent to $\Delta u \geq 0$ at every point.*

Proof. Taylor expansion at $a = (a_1, a_2)$, with $e_t = (\cos t, \sin t)$, gives

$$u(a + re_t) = u(a) + r\nabla u(a) \cdot e_t + \frac{r^2}{2} e_t^T D^2 u(a) e_t + o(r^2).$$

The remainder is uniform in t because the Hessian is continuous near a. The averages of $\cos t$, $\sin t$, and $\sin t \cos t$ vanish, whereas the averages of $\cos^2 t$ and $\sin^2 t$ are $1/2$. Hence

$$M_u(a, r) - u(a) = \frac{r^2}{4} \Delta u(a) + o(r^2).$$

The submean inequality, divided by r^2 and followed by $r \downarrow 0$, implies $\Delta u(a) \geq 0$. For the converse differentiate the circular mean for $r > 0$ and use the planar divergence theorem on $D(a, r)$:

$$M_u'(a, r) = \frac{1}{2\pi} \int_0^{2\pi} \partial_r u(a + re^{it}) \, \mathrm{d}t = \frac{1}{2\pi r} \int_{D(a,r)} \Delta u \, dA \geq 0.$$

Continuity gives $M_u(a, r) \to u(a)$ as $r \downarrow 0$. Integrating the nonnegative derivative proves the required inequality for every admissible radius, not just infinitesimal radii. □

In particular $|z|^2$ is subharmonic because $\Delta |z|^2 = 4$. It has a strict minimum at zero. The subharmonic theory therefore has no minimum principle of the harmonic kind. Conversely $-|z|^2$ is superharmonic, not subharmonic. The sign of the Laplacian determines which comparison direction is valid.

6.2 The maximum principle with its exact quantifier

Theorem 6.3 (Strong and boundary maximum principles)**.** *If a subharmonic function on connected Ω attains its finite global supremum at an interior point, it is constant. If Ω is bounded and*

$$\limsup_{\Omega \ni z \to \xi} u(z) \leq 0 \quad \text{for every } \xi \in \partial\Omega,$$

then $u \leq 0$ throughout Ω.

Proof. First suppose $u \leq M$ and $u(a) = M$. On each sufficiently small circle centered at a, the submean inequality says

$$M \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{it}) \, \mathrm{d}t \leq M.$$

If $u(b) < M$ at any point of that circle, upper semicontinuity supplies an open arc on which $u \leq M - \eta$ for some $\eta > 0$. Its positive angular length would make the average strictly less than M. Thus the entire circle has value M. Allowing every small radius shows that $u = M$ on a disk about a. The set $E = \{u = M\}$ is open by this argument and closed because $\{u < M\}$ is open. Connectedness and $E \neq \varnothing$ give $E = \Omega$.

For the boundary statement suppose $u(a) > 0$, and fix $0 < \varepsilon < u(a)$. At every boundary point the limsup hypothesis gives an open neighborhood on which $u < \varepsilon$ inside Ω. Finitely many such

neighborhoods cover the compact boundary. Their complement in $\overline{\Omega}$ is a compact subset $K \subset \Omega$; it contains a. Upper semicontinuity gives a maximum on K, and this is a global maximum of u because all points outside K have value less than ε. The first part forces u to be a positive constant, contradicting the boundary limsup. This proves the result without smoothness, local connectedness, or accessibility assumptions on $\partial\Omega$. □

The word *global* cannot be dropped. The function

$$u(z) = \log^+ |z| := \max\{0, \log |z|\}$$

is constant on $D(0,1)$ and nonconstant on the plane. Every point of that open disk is a local maximum in the non-strict sense. This corrects the local-maximum wording in the supplied source's subharmonic maximum principle. A strict interior local maximum is impossible; a flat local maximum does not force a global constant. The distinction is essential when harmonic replacements create flat portions of a comparison function.

Theorem 6.4 (Boundary comparison)**.** *Let G be a bounded domain. Suppose u is subharmonic on G, h is harmonic there, and*

$$\limsup_{G \ni z \to \xi} \big(u(z) - h(z)\big) \le 0 \quad (\xi \in \partial G).$$

Then $u \le h$ on G. The same conclusion holds with a finite continuous superharmonic function h in place of the harmonic function.

Proof. In the harmonic case, $w = u - h$ is upper semicontinuous and

$$w(a) = u(a) - h(a) \le M_u(a, r) - M_h(a, r) = M_w(a, r).$$

It is nontrivial because h is finite and u is finite somewhere. Apply the boundary maximum principle to w. In the superharmonic case the inequality $h(a) \ge M_h(a, r)$ gives the same chain. Continuity and finiteness of h ensure that subtraction preserves upper semicontinuity and avoids undefined extended values. □

Source correspondence. Lyubich, Lemma 10.15, with the roles corrected: the lower function is subharmonic and the upper comparator is harmonic.

The role reversal in the printed lemma would assert, for example, that $0 \le |z|^2 - 1$ in the unit disk because equality holds on its boundary. That conclusion is false at zero. The corrected theorem has the exact inequality required by the proof and by the later Perron construction.

6.3 Local information and comparison determine all means

Proposition 6.5 (Local means and continuous comparison)**.** *For an upper semicontinuous nontrivial u, it is enough to require the submean inequality for all sufficiently small radii at each point. For a finite continuous function u, it is also equivalent to require harmonic comparison on every relatively compact disk: if h is harmonic in the disk, continuous on its closure, and $u \le h$ on its circle, then $u \le h$ in the disk.*

Proof. The proof of the strong maximum principle used only small circles; the boundary proof therefore remains valid under the local hypothesis. Consequently harmonic comparison holds on every relatively compact disk. For continuous u, solve its disk Dirichlet problem by Theorem 3.2. Comparison with that solution at the center is exactly the full-radius submean inequality.

For general upper semicontinuous u, approximate its restriction b to a compact circle K from above by continuous functions. Here is a concrete construction which also handles $-\infty$ values:

$$b_n(x) = \sup_{y\in K}\{\max(b(y), -n) - n|x-y|\}, \qquad x \in K.$$

Each b_n is finite and n-Lipschitz, $b_n \geq b$, and $b_{n+1} \leq b_n$. To prove $b_n(x) \downarrow b(x)$, fix a finite threshold $c > b(x)$. Upper semicontinuity gives $\delta > 0$ such that $b(y) < c$ when $|x-y| < \delta$. If B bounds b above, then for large n

$$|x-y| \geq \delta \Longrightarrow \max(b(y), -n) - n|x-y| \leq \max(B, -n) - n\delta < c.$$

For $|x-y| < \delta$, both $b(y)$ and $-n$ are eventually less than c. Thus $b_n(x) \leq c$ eventually. Arbitrary thresholds $c > b(x)$ prove the claim, also when $b(x) = -\infty$. The inequalities in this construction can be tracked without selecting a maximizing point. For $x, x' \in K$ and every $y \in K$,

$$\max(b(y), -n) - n|x-y| \leq \max(b(y), -n) - n|x'-y| + n|x-x'|.$$

Taking suprema and exchanging x, x' proves the Lipschitz estimate. Also $\max(b(y), -(n+1)) \leq \max(b(y), -n)$ and $-(n+1)|x-y| \leq -n|x-y|$, proving monotonicity before any limit is taken. The choice $y = x$ gives $b_n(x) \geq b(x)$, including infinite values. For a finite upper bound B, choose n so that $-n < c$ and $\max(B, -n) - n\delta < c$. Splitting the supremum into the two sets $|y-x| < \delta$ and $|y-x| \geq \delta$ then bounds every term by c. The conclusion is $b_n(x) \leq c$; a supremum of strictly smaller terms need not itself be strictly smaller. This distinction avoids an unnecessary compactness or attainment assumption.

If b is continuous, the convergence is uniform. To see the additional hypothesis at work, choose one δ from uniform continuity so that $|b(y) - b(x)| < \eta$ whenever $|x-y| < \delta$. The finite minimum of b allows $-n < \min_K b$, and $n\delta > B - \min_K b$ excludes the distant terms. Consequently

$$b(x) \leq b_n(x) \leq b(x) + \eta \quad (x \in K)$$

for all sufficiently large n. For a general upper semicontinuous datum only pointwise monotone convergence has been proved, and that is precisely why the subsequent integral passage uses a measure theorem.

Let H_n be the Poisson extension of b_n. Boundary comparison yields $u \leq H_n$. At the center,

$$u(a) \leq H_n(a) = \frac{1}{2\pi}\int_K b_n\, dt \downarrow \frac{1}{2\pi}\int_K b\, dt.$$

The last step applies monotone convergence to $B_1 - b_n \geq 0$, where B_1 bounds b_1 above. Hence all the required circle inequalities hold. This also proves that subharmonicity is a local property. □

The same proof gives a useful stronger inequality at every interior point of $D(a, R)$:

$$u(z) \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - |z-a|^2}{|a + Re^{it} - z|^2}\, u(a + Re^{it})\, \mathrm{d}t. \tag{6.1}$$

The kernel is strictly positive and bounded above and below when z stays in a smaller concentric disk. This weighted form will control circular integrals even when the center itself is a logarithmic pole.

The constants in weighted comparison. For $|z-a| \leq qR$, $0 \leq q < 1$, the Poisson weight in (6.1) satisfies

$$c_q := \frac{1-q}{1+q} \leq \frac{R^2 - |z-a|^2}{|a + Re^{it} - z|^2} \leq \frac{1+q}{1-q} =: C_q.$$

Indeed, at fixed $\rho = |z-a|/R$ its denominator lies between $(1-\rho)^2R^2$ and $(1+\rho)^2R^2$, and its numerator is $(1-\rho^2)R^2$. This gives the two ratios before replacing ρ by q. Let B be an upper bound on the boundary datum and write $d(t) = B - u(a+Re^{it}) \geq 0$. Because the Poisson kernel has mass one, weighted comparison is equivalent to

$$\frac{1}{2\pi}\int P(z,t)d(t)\,\mathrm{d}t \leq B - u(z).$$

At any interior point where $u(z)$ is finite, it follows that

$$\frac{1}{2\pi}\int d(t)\,\mathrm{d}t \leq c_q^{-1}(B-u(z)) < \infty.$$

Thus one finite interior value controls the unweighted boundary deficit, even when some boundary values are $-\infty$. This argument does not assert a finite lower bound at every boundary point. Conversely, once the unweighted deficit is integrable, every weighted integral at a point of the smaller disk is finite and bounded by C_q times that deficit. The factor deteriorates as $q \uparrow 1$, recording why this interior estimate alone supplies no uniform boundary regularity.

6.4 Local integrability and canonical values

Theorem 6.6 (No hidden set of area poles)**.** *Every subharmonic function on a connected domain belongs to L^1_{loc}. In particular it is finite almost everywhere. Its value at every point is determined by its area averages:*

$$u(a) = \lim_{r\downarrow 0}\frac{1}{\pi r^2}\int_{D(a,r)} u\,dA.$$

Thus two subharmonic functions equal almost everywhere are equal everywhere.

Proof. Choose a with $u(a) > -\infty$ and a closed disk on which $u \leq B$. For every smaller radius the submean inequality gives

$$0 \leq \frac{1}{2\pi}\int_0^{2\pi}(B - u(a+re^{it}))\,\mathrm{d}t \leq B - u(a).$$

Integrate against $2\pi r\,dr$. Tonelli applies to the nonnegative integrand and gives

$$\int_{D(a,R)}(B-u)\,dA \leq \pi R^2(B-u(a)) < \infty.$$

Hence u is integrable near at least one point. Let F be the set of points with an integrable neighborhood. It is open and nonempty. To see it is relatively closed, take $b \in \overline{F}\cap\Omega$ and choose $R>0$ with $\overline{D(b,3R)} \subset \Omega$. There is a point of F inside $D(b,R/2)$, and an open neighborhood of that point is integrable. It therefore contains a point $a \in D(b,R)$ with finite $u(a)$. The preceding argument on $D(a,2R)$ proves integrability on a neighborhood of b. Thus F is closed; connectedness gives $F=\Omega$. A compact set has a finite cover by these integrable neighborhoods.

The disk containments in that propagation step are strict and quantitative:

$$D(b,R) \subset D(a,2R) \subset D(b,3R) \quad \text{when } |a-b| < R.$$

The first inclusion follows from $|z-a| \leq |z-b| + |b-a| < 2R$; the second follows by the same triangle inequality. To select a rigorously, intersect the integrable neighborhood with $D(b,R)$. This is a nonempty open set, hence has positive area. An integrable function is finite almost everywhere there, so the intersection contains a point where u is finite. A finite point is not obtained merely by knowing that the neighborhood is nonempty; positive area and local integrability supply that conclusion.

Polar integration of the circle inequality now yields the area inequality

$$u(a) \le A_r u(a) := \frac{1}{\pi r^2} \int_{D(a,r)} u \, dA.$$

Upper semicontinuity gives $A_r u(a) \le u(a) + \varepsilon$ for small r when $u(a)$ is finite. If $u(a) = -\infty$, it gives $A_r u(a) \le B$ for small r for every real B. These inequalities prove the stated limit in both cases. Almost-everywhere equality makes all area averages equal, and the limit recovers equality at the exceptional points as well. □

A finite chain with a visible propagation constant. The open-and-closed proof can also be made quantitative along a chosen chain. Suppose $c_0, \dots, c_N$ satisfy $|c_{j+1} - c_j| \le r/2$, every closed disk $\overline{D(c_j, 2r)}$ lies in Ω, and $u \le B$ on their union. Put

$$I_j = \int_{D(c_j,r)} (B - u) \, dA.$$

Assume $I_0 < \infty$. The small disk $E_j = D(c_{j+1}, r/4)$ is contained in $D(c_j, r)$, so its area is $\pi r^2/16$ and $\int_{E_j}(B-u) \le I_j$. If $I_j > 0$, there is a finite-valued point $a_j \in E_j$ with

$$B - u(a_j) \le \frac{2I_j}{|E_j|} = \frac{32 I_j}{\pi r^2}.$$

Otherwise the integral over E_j would exceed $2I_j$. The disk of radius $5r/4$ about a_j contains $D(c_{j+1}, r)$ and lies in $D(c_{j+1}, 3r/2)$. Applying the finite-center estimate there gives

$$I_{j+1} \le \pi(5r/4)^2 (B - u(a_j)) \le 50 I_j, \qquad I_N \le 50^N I_0.$$

If $I_j = 0$, choose a_j in the full-measure set where $B - u = 0$; the same argument gives $I_{j+1} = 0$. The bound is intentionally explicit rather than optimal. It depends on the number and geometry of the disks, and therefore makes no unsupported uniform claim as a chain approaches the boundary. Finally $|u| \le (B - u) + |B|$ when $u \le B$, so

$$\int_{D(c_N,r)} |u| \, dA \le 50^N I_0 + \pi r^2 |B|.$$

This is the mechanism behind propagation of integrability: an area bound selects a useful finite center, and its submean inequality controls a larger disk.

Investigation 6.7 (An upward spike)**.** Set $u(0) = 1$ and $u(z) = 0$ elsewhere. Decide which condition fails.

Solution. The function is upper semicontinuous: at zero its limsup is zero, and away from zero it is locally zero. But for every $r > 0$, $u(0) = 1 > M_u(0, r) = 0$. Changing instead the value at zero to -1 satisfies that particular mean inequality but fails upper semicontinuity at zero. The two requirements jointly exclude both arbitrary alterations.

Investigation 6.8 (Quadratic forms)**.** For real a, b, c, classify $u(x, y) = ax^2 + 2bxy + cy^2$.

Solution. Since $\Delta u = 2(a + c)$, the function is subharmonic exactly when $a + c \ge 0$, harmonic exactly when $a + c = 0$, and superharmonic exactly when $a + c \le 0$. No positivity of the full Hessian is required. For example $x^2 - y^2$ is harmonic, although it is neither convex nor concave. Its circular mean equals its center value because the two quadratic contributions cancel exactly.

Investigation 6.9 (A quantitative area estimate)**.** Suppose u is subharmonic on a neighborhood of $\overline{D(a, R)}$, $u \le B$ there, and $u(a) \ge -B$ with $B \ge 0$. Bound $\int |u|$.

Solution. The proof above gives $\int(B-u) \le 2\pi R^2 B$. Since $|u| \le (B-u)+B$ whenever $u \le B$,

$$\int_{D(a,R)} |u|\, dA \le 3\pi R^2 B.$$

The statement uses a finite lower bound at one center, not a lower bound on every point. It is a local integrability estimate rather than a bound on the minimum of u.

Investigation 6.10 (Why boundedness of the domain matters)**.** Give a harmonic function on the upper half-plane with boundary values zero which is positive inside.

Solution. The function $h(x+iy) = y$ has $\Delta h = 0$, vanishes continuously on $\mathbb{R}$, and is positive for $y > 0$. Thus a boundary inequality on the finite boundary alone does not imply comparison in an unbounded domain. In the bounded-domain proof, compactness prevented the positive maximum from escaping to infinity; here $h(iy) = y \to \infty$.

Investigation 6.11 (A comparison counterexample)**.** Test the statement "a harmonic function below a subharmonic function on the boundary stays below it inside" on the unit disk.

Solution. Take $h = 0$ and $u = |z|^2 - 1$. On $|z| = 1$ they agree, but $h(0) = 0 > -1 = u(0)$. The valid conclusion is the reverse ordering $u \le h$, because $\Delta u = 4 \ge 0$. This single exact example detects the role reversal independently of any interpretation of the source.

Investigation 6.12 (Canonical representatives)**.** Suppose u, v are subharmonic and $u = v$ outside a set of area zero. Can they differ at one point where both are finite?

Solution. No. For every sufficiently small disk their integrals agree, so $A_r u(a) = A_r v(a)$. Taking $r \downarrow 0$ in Theorem 6.6 gives $u(a) = v(a)$, including any infinite values. Upper semicontinuity alone would not give this conclusion, as the upward-spike example shows; the area inequalities are the missing constraint.

Chapter 7

Stability, harmonic lifting, and removable punctures

Perron's construction needs competitors which can be raised locally while remaining admissible. Three elementary operations provide them: taking a finite maximum, pasting a larger competitor across a controlled seam, and replacing a disk by the harmonic function with the same boundary data. Their proofs must examine the seam. Separate harmonicity on the two open sides does not by itself determine the sign of the Laplacian on their common boundary.

7.1 Operations that preserve the inequality

Proposition 7.1 (Positive sums and finite maxima)**.** *Finite positive linear combinations of subharmonic functions are subharmonic. The maximum of finitely many subharmonic functions is subharmonic. Adding a harmonic function preserves subharmonicity.*

Proof. Local integrability makes every summand finite outside an area-zero set; a finite sum is therefore finite almost everywhere. Upper semicontinuity survives a positive finite sum because each summand is locally bounded above and no $+\infty$ value occurs. Adding the mean inequalities proves the assertion. A coefficient zero means that the corresponding term is omitted, so $0(-\infty)$ is never evaluated. For $w = \max(u_1, \dots, u_m)$, upper semicontinuity follows from

$$\{w < c\} = \bigcap_{j=1}^{m} \{u_j < c\}.$$

At a point where $w(a)$ is finite choose an index j with $w(a) = u_j(a)$. Then

$$w(a) = u_j(a) \le M_{u_j}(a, r) \le M_w(a, r).$$

At a point with $w(a) = -\infty$ the inequality is automatic. The statement for adding a harmonic function follows by adding its exact mean identity. □

Proposition 7.2 (Decreasing limits)**.** *If u_n are subharmonic on connected Ω and $u_n \downarrow u$ pointwise, then either $u \equiv -\infty$ or u is subharmonic.*

Proof. The infimum of a decreasing sequence of upper semicontinuous functions is upper semicontinuous, since $\{\inf_n u_n < c\} = \bigcup_n \{u_n < c\}$. On any compact circle choose B bounding u_1 above. The nonnegative functions $B - u_n$ increase to $B - u$, so monotone convergence gives $M_{u_n}(a, r) \downarrow M_u(a, r)$. Passing to the limit in $u_n(a) \le M_{u_n}(a, r)$ proves the mean inequality. Nontriviality is the only remaining condition, and it is explicitly assumed in the second alternative. □

An arbitrary increasing limit need not be upper semicontinuous. For $u_n(z) = \max\{n \log |z|, -1\}$ on $\mathbb{D}$, one has $u_n \downarrow -1$ at every point; this is harmless. For the increasing sequence $\max\{n^{-1} \log |z|, -1\}$, the pointwise limit is zero away from zero and -1 at zero. It fails upper semicontinuity. A pointwise supremum used in Perron's method therefore needs a separate proof of harmonicity; it cannot be accepted from monotonicity alone.

Proposition 7.3 (Convex increasing functions)**.** *Let u be finite continuous subharmonic and let $\Phi : \mathbb{R} \to \mathbb{R}$ be convex and nondecreasing. Then $\Phi \circ u$ is subharmonic.*

Proof. A finite convex function on $\mathbb{R}$ is continuous, so the composition is continuous. On each circle, monotonicity followed by Jensen's inequality for the probability measure $dt/(2\pi)$ gives

$$\Phi(u(a)) \le \Phi(M_u(a, r)) \le \frac{1}{2\pi} \int_0^{2\pi} \Phi(u(a + re^{it}))\, \mathrm{d}t.$$

For completeness, the scalar Jensen inequality follows by choosing a supporting affine line $\ell(t) = \Phi(m) + p(t - m)$ at the mean m. Convexity gives $\Phi(t) \ge \ell(t)$; integration cancels the term $p(t - m)$. A supporting slope exists because the left derivative does not exceed the right derivative of a convex function. All circle values here lie in a compact interval, so every integral is finite. □

For smooth functions the same preservation law is visible in

$$\Delta\Phi(u) = \Phi'(u)\Delta u + \Phi''(u)|\nabla u|^2 \ge 0.$$

The mean proof also covers corners in Φ, such as $\Phi(t) = \max(t, c)$, and avoids assuming differentiability of u.

7.2 Pasting and disk replacement

Theorem 7.4 (Pasting across a seam)**.** *Let $G \subset \Omega$ be open, let u be subharmonic on Ω, and let v be subharmonic on G. Suppose*

$$\limsup_{G \ni z \to \xi} v(z) \le u(\xi) \quad (\xi \in \partial G \cap \Omega).$$

Then the function

$$w(z) = \begin{cases} \max\{u(z), v(z)\}, & z \in G, \\ u(z), & z \notin G \end{cases}$$

is subharmonic on Ω.

Proof. Away from the seam the claim is local and follows from finite-max stability. At a seam point ξ, the assumed limsup and upper semicontinuity of u show $\limsup_{z \to \xi} w(z) \le u(\xi) = w(\xi)$. For clarity, the seam argument also covers $u(\xi) = -\infty$. Given any real A, upper semicontinuity supplies a neighborhood on which $u < A$, and the seam hypothesis supplies a possibly smaller neighborhood on whose intersection with G one has $v < A$. Hence $w < A$ throughout their intersection. This is exactly upper semicontinuity with infinite value. When $u(\xi)$ is finite, replace A by $u(\xi) + \eta$ and then let $\eta \downarrow 0$. The approaches from inside and outside G are both included; no smoothness or local connectedness of the seam is required.

Thus w is upper semicontinuous everywhere. On any small circle centered at a seam point,

$$w(\xi) = u(\xi) \le M_u(\xi, r) \le M_w(\xi, r),$$

because $w \geq u$ throughout Ω. Hence the local submean condition holds at seam points as well. Proposition 6.5 now gives the full subharmonic condition. Since $w \geq u$, it cannot be identically $-\infty$. This proof explains precisely why taking the maximum on the inside is part of the construction. □

Theorem 7.5 (Harmonic disk lifting)**.** *Let u be finite continuous subharmonic on Ω, and let $\overline{D} = \overline{D(a,R)} \subset \Omega$. Let $H_D u$ be the harmonic extension of $u|_{\partial D}$. Define*

$$L_D u(z) = \begin{cases} H_D u(z), & z \in D, \\ u(z), & z \in \Omega \setminus D. \end{cases}$$

Then $L_D u$ is finite continuous subharmonic, $u \leq L_D u$, and $L_D u = u$ off D. Lifting is order preserving and idempotent:

$$u \leq v \Longrightarrow L_D u \leq L_D v, \qquad L_D(L_D u) = L_D u.$$

Proof. The disk Dirichlet theorem gives a harmonic $H_D u$ continuous on the closed disk, with the specified boundary values. Comparison gives $u \leq H_D u$ in D. The boundary values match, so the piecewise function is continuous. Apply the pasting theorem with $v = H_D u$; because $v \geq u$ on D, its pasted maximum is exactly $L_D u$. If $u \leq v$, positivity of the Poisson kernel gives

$$H_D u(z) \leq H_D v(z) \quad (z \in D),$$

and outside D the original inequality remains. Finally the first lift has the same boundary trace as u, so its second harmonic replacement is the same unique disk solution. This proves idempotence. □

The lift is also the least harmonic majorant with this boundary trace: any harmonic function continuous on $\overline{D}$ with boundary values at least u dominates $H_D u$ by harmonic comparison. In particular, a uniform upper bound on u is preserved by lifting. If $u \leq B$, the constant B is such a harmonic majorant and $L_D u \leq B$.

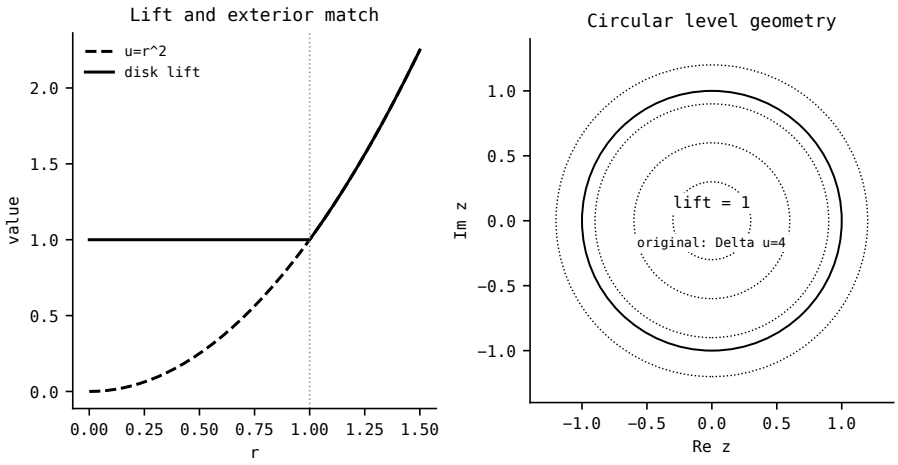


Figure 7.1: An exact disk lift. For $u(z) = |z|^2$ and the unit disk, the lift is 1 inside and $|z|^2$ outside. The original and lifted functions agree on the circle, while the lifted graph has a flat interior. Line styles distinguish the functions; the seam carries positive Laplacian mass rather than a violation of subharmonicity.

7.3 Where the source mass goes during a lift

Pasting requires no derivatives at the seam, but a smooth example reveals what its inequality means. Let a smooth curve Γ separate an inside region from an outside region, and let n point from inside to outside. Suppose w_- and w_+ are twice continuously differentiable up to their respective sides and have the same trace on Γ. For every smooth compactly supported test function φ, integrate twice by parts on each side. The inside boundary term is $w_-\partial_n\varphi - \varphi\partial_n w_-$; the outside normal on the same curve is $-n$, so its term is $-w_+\partial_n\varphi + \varphi\partial_n w_+$. The common trace cancels the derivatives of the test function. Thus

$$\int w\Delta\varphi\, dA = \int_{\text{inside}} \varphi\Delta w_-\, dA + \int_{\text{outside}} \varphi\Delta w_+\, dA + \int_\Gamma \varphi(\partial_n w_+ - \partial_n w_-)\, ds.$$

This identity defines the distributional source without presuming the later distributional characterization theorem. It separates ordinary area density from a source concentrated on the seam. The sign of the jump uses one fixed normal on both derivatives; using each region's outward normal inside the difference would reverse one term incorrectly.

For a nonradial example fix $\beta \in \mathbb{R}$ and set

$$u(r,\theta) = r^2 + \beta r^3\cos 3\theta, \qquad L(r,\theta) = \max\{1, r^2\} + \beta r^3\cos 3\theta.$$

The cubic term is $\beta\,\mathrm{Re}(z^3)$ and is harmonic. On the unit circle the boundary trace is $1+\beta\cos 3\theta$, whose harmonic extension is $1+\beta r^3\cos 3\theta$. Therefore $L = L_{\mathbb{D}}u$ by the already proved lifting theorem. Direct differentiation gives

$$\partial_r L|_{1-} = 3\beta\cos 3\theta, \qquad \partial_r L|_{1+} = 2 + 3\beta\cos 3\theta, \qquad [\partial_r L] = 2.$$

The derivatives can change sign with angle, but their difference is positive everywhere. Their common angular variation changes level curves without changing the source. The test identity becomes

$$\Delta L = 4\mathbf{1}_{\{|z|>1\}}\, dA + 2\, ds|_{\{|z|=1\}}, \qquad \Delta u = 4\, dA.$$

Writing $\mu = (2\pi)^{-1}\Delta L$ in the normalization introduced below, the seam has total mass $2(2\pi)/(2\pi) = 2$. For closed centered disks,

$$\mu(\overline{D(0,r)}) = \begin{cases} 0, & 0 \le r < 1, \\ 2r^2, & r \ge 1. \end{cases}$$

For $r > 1$, the calculation is $2 + \frac{2}{\pi}\pi(r^2-1) = 2r^2$. The original source also assigns mass $2r^2$ to the radius-r disk. Hence lifting removes the interior area source and places exactly the removed mass on the boundary, while every larger centered disk retains its original total. At $r = 1$ the open disk has zero lifted mass and the closed disk has mass two. That distinction will reappear as the left/right derivative distinction in Jensen's formula.

7.4 An isolated singularity bounded from above

Theorem 7.6 (Removable subharmonic puncture)**.** *Let u be subharmonic on $D(a,R)\setminus\{a\}$ and bounded above near a. Define*

$$U(z) = u(z)\ (z \ne a), \qquad U(a) = \limsup_{z\to a} u(z) \in [-\infty, \infty).$$

Then U is subharmonic on $D(a,R)$. It is the unique subharmonic extension across a.

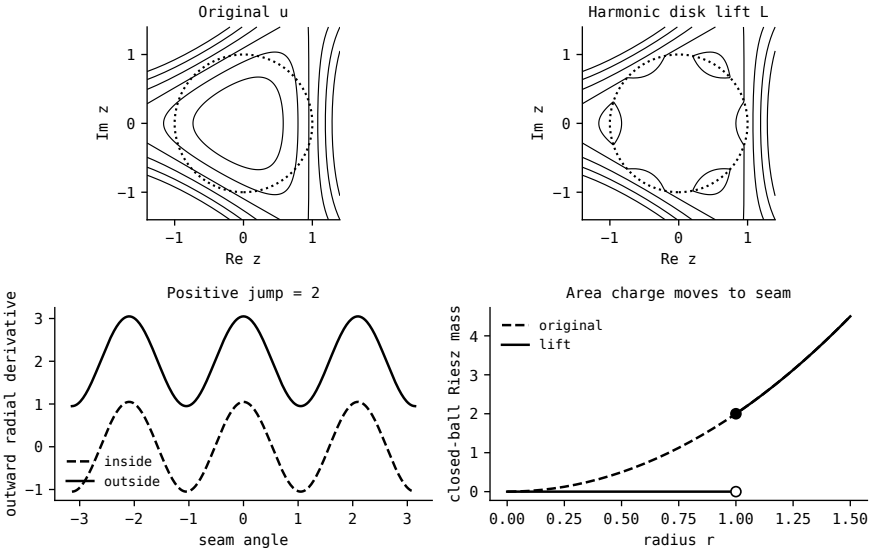


Figure 7.2: A nonradial lift with $\beta = 0.35$. The cubic harmonic component changes the contour geometry while leaving $\Delta u = 4$. The two angular normal derivatives differ by exactly two; the lifted source moves the disk's mass onto its unit-circle seam. The cumulative source plot uses closed disks, with the value at the jump distinguished from its left limit.

Proof. The limsup definition makes U upper semicontinuous at a and leaves it unchanged elsewhere. To verify the submean inequality at a, fix $0 < r < R$ and continuous functions $b_n \downarrow u|_{\partial D(a,r)}$ as constructed in Proposition 6.5. Write H_n for their harmonic disk extensions. For any $\varepsilon > 0$ consider

$$w_{n,\varepsilon}(z) = u(z) - H_n(z) + \varepsilon \log \frac{|z-a|}{r} \quad (0 < |z-a| < r).$$

This is subharmonic on the punctured disk. Its boundary limsup on the outer circle is at most zero. As $z \to a$, the bounded-above hypothesis and boundedness of H_n imply $w_{n,\varepsilon}(z) \to -\infty$. For a fixed n, let C bound u near the puncture and let $m_n = \min_{\overline{D(a,r)}} H_n$, which is finite. On a sufficiently small circle of radius ρ,

$$w_{n,\varepsilon} \le C - m_n + \varepsilon \log(\rho/r).$$

The explicit condition $\rho < r \exp[-(\max\{C - m_n, 0\} + 1)/\varepsilon]$, together with the radius where the bound C holds, makes this less than zero. For a fixed interior $z \neq a$, choose also $\rho < |z-a|$. The small radius is allowed to depend on n and ε; no estimate uniform in both parameters is needed. The successive limits keep z fixed, first remove the artificial inner circle, then send ε to zero, and only afterwards approximate the outer boundary datum.

Hence on every sufficiently small inner circle it is negative. Boundary comparison on the annulus between that circle and $\partial D(a,r)$ gives $w_{n,\varepsilon} \le 0$. For each fixed $z \neq a$, let the inner radius decrease to zero and then let $\varepsilon \downarrow 0$ to obtain $u(z) \le H_n(z)$. Taking limsup at a gives

$$U(a) \le H_n(a) = \frac{1}{2\pi} \int_0^{2\pi} b_n(a + re^{it})\, \mathrm{d}t.$$

Monotone convergence yields $U(a) \le M_U(a,r)$. At other points the local submean inequality was already known, so Proposition 6.5 proves subharmonicity on the whole disk. If another subharmonic extension exists, it agrees with U almost everywhere. The canonical-value theorem makes it equal everywhere. □

The conclusion permits $U(a) = -\infty$. Thus "removable" refers to extension in the subharmonic class, not necessarily to extension as a finite continuous or harmonic function. For $u = \log|z-a|$, the extension is subharmonic with a logarithmic pole. For $u = -\log|z-a|$, local boundedness above fails, and no finite-valued-above subharmonic extension exists.

7.5 Coordinate changes

For a biholomorphic map $\phi : V \to \Omega$ and a continuous subharmonic u, the composition $u \circ \phi$ is subharmonic. To see this without assuming u smooth, choose a relatively compact disk $D \subset V$ and a harmonic comparator h with $u \circ \phi \leq h$ on ∂D. The function $h \circ \phi^{-1}$ is harmonic on $\phi(D)$ by conformal invariance of harmonicity, and the boundary inequality transports there. Boundary comparison gives $u \leq h \circ \phi^{-1}$ on $\phi(D)$, hence $u \circ \phi \leq h$ on D. The continuous comparison characterization finishes the proof. General subharmonic functions, and holomorphic maps with critical points, will be treated after smooth approximation below.

Investigation 7.7 (Minimum is different from maximum)**.** Show that $\min\{x, -x\}$ is not subharmonic on $\mathbb{C}$.

Solution. Both x and $-x$ are harmonic. Their minimum is $-|x|$, whose value at zero is zero, but on the circle re^{it} its mean is

$$-\frac{r}{2\pi}\int_0^{2\pi} |\cos t|\,\mathrm{d}t = -\frac{2r}{\pi} < 0.$$

Thus the submean inequality fails. The maximum $|x|$, in contrast, is subharmonic by finite-max stability. The crease orientation matters.

Investigation 7.8 (Compute a lift)**.** Lift $u(z) = |z|^2 + \mathrm{Re}(z^3)$ in $D(0, R)$.

Solution. On the boundary, $|z|^2 = R^2$, so the unique harmonic extension is $H(z) = R^2 + \mathrm{Re}(z^3)$. Consequently

$$H(z) - u(z) = R^2 - |z|^2 \geq 0 \quad (|z| \leq R).$$

Outside the disk the lift leaves u unchanged. The harmonic cubic does not affect the gap; lifting removes precisely the quadratic mean defect.

Investigation 7.9 (A failed seam)**.** Extend $1 - |z|^2$ by zero outside the unit disk. Is the result subharmonic?

Solution. No: on the open disk its Laplacian is -4, already violating the smooth criterion. Even the piecewise harmonic function $\min(0, \log|z|)$ fails the correct seam direction. Its radial derivative drops from 1 at the inner side of $r = 1$ to 0 at the outer side, corresponding to negative boundary mass. The pasting theorem uses an inside maximum and a compatible limsup, neither of which licenses this minimum construction.

Investigation 7.10 (Puncture uniqueness)**.** Can $\log|z|$ on $\mathbb{D} \setminus \{0\}$ be extended subharmonically with $u(0) = 0$?

Solution. On every circle of radius $r < 1$, the mean is $\log r < 0$. The proposed value would require $0 \leq \log r$, a contradiction. The canonical extension has $u(0) = -\infty$. Boundedness above near the puncture guarantees the extension theorem, but does not prescribe a finite value there.

Investigation 7.11 (Convexity and a power)**.** Prove that $|f|^p$ is subharmonic for a holomorphic f and every $p > 0$.

Solution. Away from zeros, $\log|f|$ is harmonic and $\Phi(t) = e^{pt}$ is increasing and convex. Thus $|f|^p$ is subharmonic there. It extends continuously by zero across each zero, is locally bounded above, and the removable-puncture theorem gives subharmonicity at the zeros. The identically zero function also satisfies the submean inequality directly. This argument works for $0 < p < 1$, although $t \mapsto t^p$ is concave on positive real numbers.

Investigation 7.12 (Increasing limits need repair)**.** Find the upper semicontinuous regularization of $u(z) = \lim_n \max(n^{-1}\log|z|, -1)$ on $\mathbb{D}$.

Solution. The pointwise limit is 0 for $z \neq 0$ and -1 at zero. Its upper regularization $u^*(a) = \limsup_{z\to a} u(z)$ is identically zero. Hence the limit and its regularization differ at precisely one point. This example warns against passing an arbitrary pointwise increasing limit directly into a theorem which requires upper semicontinuity.

Chapter 8

Logarithmic singularities, Jensen's formula, and zero counts

A zero of a holomorphic function becomes a negative logarithmic singularity of its modulus. Averaging the logarithm around a circle records how many zeros have entered the circle, with their multiplicities. The basic identity is particularly informative because it separates the behavior inside, on, and outside the averaging circle; the on-circle case is integrable but not a consequence of uniform continuity of the logarithm.

8.1 The exact circular mean of one logarithm

Lemma 8.1 (A translated logarithm)**.** *For $a, b \in \mathbb{C}$ and $r > 0$,*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|a + re^{it} - b|\,\mathrm{d}t = \log\max\{r, |a-b|\}.$$

If $|a-b| = r$, the integral is a finite Lebesgue integral despite its single logarithmic singularity.

Proof. After translation and rotation it suffices to study $\log|re^{it} - d|$ for $d \geq 0$. If $d < r$, write

$$\log|re^{it} - d| = \log r + \operatorname{Re}\log(1 - (d/r)e^{-it}).$$

The logarithm is given by the uniformly convergent series $-\sum_{n\geq 1}(d/r)^n e^{-int}/n$. Every nonconstant Fourier mode has mean zero, leaving $\log r$. If $d > r$, factor out d instead and apply the same calculation with $r/d < 1$, obtaining $\log d$.

For $d = r$, scaling reduces the question to $\log|e^{it} - 1|$. Represent t in $[-\pi, \pi]$. For $1/2 \leq q \leq 1$,

$$|e^{it} - q|^2 = (1-q)^2 + 4q\sin^2(t/2) \geq \frac{2t^2}{\pi^2}, \qquad |e^{it} - q| \leq 2.$$

Here $|\sin(t/2)| \geq |t|/\pi$. Thus the positive part of the logarithm is bounded by $\log 2$, and its negative part is bounded by $\max\{0, -\log(\sqrt{2}|t|/\pi)\}$, an integrable function independent of q. Dominated convergence as $q \uparrow 1$ passes the previously computed zero mean to the endpoint. Multiplying back by r adds $\log r$. The integrability at $t = 0$ can be read from an exact primitive. For $0 < \delta \leq 1$, define

$$\Gamma(t) = \log 2 + \max\{0, -\log(\sqrt{2}|t|/\pi)\}.$$

Then $|\log|e^{it} - q|| \leq \Gamma(t)$ for all $q \in [1/2, 1]$, and

$$\int_{-\delta}^{\delta} \Gamma(t)\,\mathrm{d}t = B(\delta) := 2\delta\left[\log 2 + 1 + \log\frac{\pi}{\sqrt{2}\delta}\right] \longrightarrow 0.$$

Indeed $\int_0^\delta -\log(ct)\,dt = \delta[1-\log(c\delta)]$, since $t\log t \to 0$. The same single bound controls every member of the family, rather than giving a separate integrability statement for each q. This uniform small-arc estimate is what prevents concentration at the approaching tangent point.

This argument justifies the boundary case with an explicit integrable majorant rather than an unsupported appeal to continuity. □

A rate for the tangent-circle passage. The dominated-convergence step has an explicit error decomposition. Write $g_q(t) = \log|e^{it} - q|$, $h = 1 - q$, and split the circle into $|t| < \delta$ and $|t| \geq \delta$, where $0 < \delta \leq 1$. The first piece contributes at most $2B(\delta)$ to $\|g_q - g_1\|_{L^1(-\pi,\pi)}$. On the second piece, for every $s \in [q, 1]$,

$$\left|\frac{\partial}{\partial s}\log|e^{it} - s|\right| \leq \frac{1}{|e^{it} - s|} \leq \frac{\pi}{\sqrt{2}\delta}.$$

The real-variable mean-value estimate and the length 2π of the integration interval give

$$\|g_q - g_1\|_1 \leq 2B(\delta) + \frac{\sqrt{2}\pi^2 h}{\delta}.$$

Choosing $\delta = \sqrt{h}$ for $0 < h \leq 1/2$ proves an explicit bound tending to zero. It is not an optimal rate, but it is derived on the whole circle and includes the singular region. In particular the absolute error in the normalized circular mean is at most this bound divided by 2π. A uniform pointwise estimate on the entire circle would be false, since $g_1(0) = -\infty$ while $g_q(0) = \log(1-q)$ is finite for $q < 1$. The decomposition explains why an integral limit remains legitimate when uniform convergence fails at the tangent point.

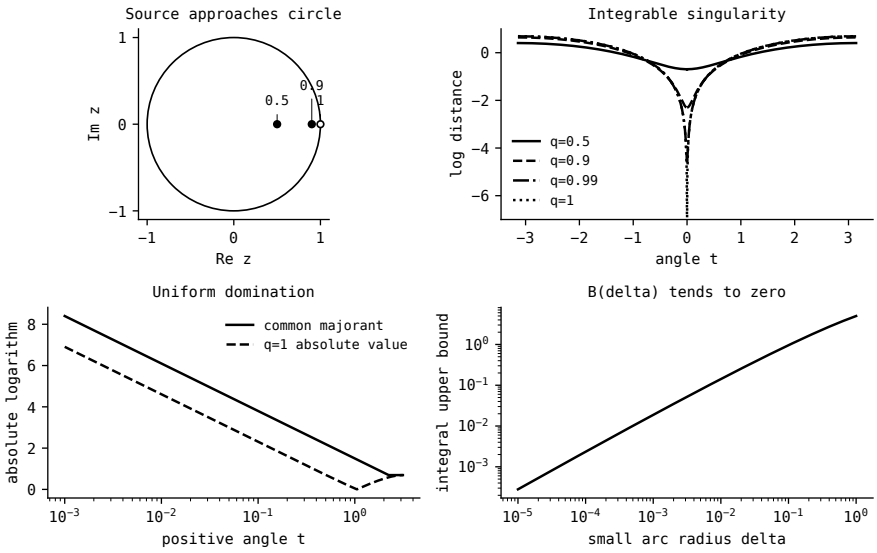


Figure 8.1: A source approaching the unit circle. The angular logarithms are controlled by one integrable majorant, whose small-arc integral is $B(\delta)$. The tangent logarithm is undefined as a finite number at $t = 0$; its curve omits that point. The error bound combines a small singular arc with an estimate on its complement, rather than replacing the singularity by a finite sampled value.

Corollary 8.2 (The basic pole)**.** *The function* $\log|z-b|$*, assigned value* $-\infty$ *at* b*, is subharmonic on* $\mathbb{C}$ *and harmonic away from* b*.*

Proof. Upper semicontinuity at the pole follows from $\log|z-b| \to -\infty$. Away from the pole a local holomorphic logarithm has this function as its real part. The mean formula gives

$$\log|a-b| \le \log\max\{r, |a-b|\}$$

when $a \neq b$, and the mean inequality at $a = b$ is automatic. It also shows precisely when the mean defect is positive: the pole lies strictly inside the averaging circle and is different from its center. □

Source correspondence. Lyubich, Example 10.13, including the tangent-circle case.

8.2 Factorization records multiplicity

Let f be holomorphic and not identically zero on a connected domain. At a zero b, its Taylor series has a first nonzero coefficient:

$$f(z) = (z-b)^m g(z), \qquad m \in \mathbb{N}, \qquad g(b) \neq 0.$$

The integer m is the order of the zero. In a sufficiently small disk, g is nonvanishing, so

$$\log|f(z)| = m\log|z-b| + \log|g(z)|.$$

The second term is harmonic. This proves that $\log|f|$ is locally subharmonic, hence subharmonic on the whole domain, with value $-\infty$ at each zero. It also identifies exactly the singular coefficient. A picture showing one zero cannot distinguish a simple zero from one of order ten; the coefficient of the logarithm does.

Theorem 8.3 (Jensen's formula with boundary zeros)**.** *Let f be holomorphic on a neighborhood of $\overline{D(a,R)}$, not identically zero, and suppose $f(a) \neq 0$. Let $b_1, \dots, b_N$ be its zeros in $D(a,R)$, listed with multiplicity. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(a+Re^{it})|\,\mathrm{d}t = \log|f(a)| + \sum_{j=1}^{N} \log\frac{R}{|b_j - a|}.$$

Zeros on $\partial D(a,R)$ are allowed; they contribute integrable singularities to the left side and a term zero if included in the sum.

Proof. The zero set is discrete, so its intersection with the compact closed disk is finite. Divide f by the product of all its factors for zeros in the closed disk, including boundary zeros with their multiplicities:

$$f(z) = g(z)\prod_{j=1}^{N'}(z-b_j).$$

The compactness argument uses the neighborhood assumption at a precise point. An infinite collection of distinct zeros in the closed disk would have an accumulation point in that closed disk. Since f is holomorphic on an open neighborhood of it, the accumulation point is an interior point of the holomorphy domain, and the identity theorem would force f to be identically zero on the component containing the disk. This contradicts the hypothesis. At each of the finitely many distinct zeros b, write

$$f(z) = (z-b)^{m_b} g_b(z), \qquad g_b(b) \neq 0.$$

If $Q(z) = \prod_b (z-b)^{m_b}$, then in a small disk about b,

$$\frac{f(z)}{Q(z)} = \frac{g_b(z)}{\prod_{c\neq b}(z-c)^{m_c}}.$$

The denominator on the right is nonzero there after shrinking the disk. This gives an explicit holomorphic extension of the quotient and shows that its value at b is nonzero. At points of the closed disk not among the zeros, numerator and denominator are both nonzero. The resulting quotient is consequently nonvanishing everywhere on the closed disk. If there are no zeros, the product is one and no cancellation is required.

Continuity now gives a strictly positive minimum of $|g|$ on the closed disk. A finite open cover of the disk by neighborhoods on which $g \neq 0$ provides the surrounding zero-free neighborhood used below. A global choice of $\log g$ is not needed merely to conclude that $\log|g|$ is harmonic: around each point a local logarithm exists, and its real part is $\log|g|$. Harmonicity is local, while the disk mean-value theorem applies to this single globally defined real function. Finally each zero on the boundary contributes its own integrable logarithmic term, already controlled by the tangent-circle estimate. A finite sum of these terms is integrable even if several zeros have the same radius or a large multiplicity. This verifies the integral hypotheses before Jensen's finite subtraction is performed.

The quotients extend holomorphically across their apparent singularities, by the local Taylor factorization. The resulting g is nonzero on the closed disk and, after shrinking its surrounding neighborhood if needed, nonzero there as well. Hence $\log|g|$ is harmonic in a neighborhood of the closed disk, and its circular mean is $\log|g(a)|$. Apply Lemma 8.1 to every factor:

$$M_{\log|f|}(a,R) = \log|g(a)| + N'\log R, \qquad \log|f(a)| = \log|g(a)| + \sum_{j=1}^{N'} \log|a-b_j|.$$

Subtract the second equality from the first. Boundary factors have $|a-b_j| = R$ and cancel. Their integrability was proved in the translated logarithm lemma, so summing these equalities is legitimate. □

Source correspondence. Lyubich, Remark 10.14, with multiplicities and boundary hypotheses explicit.

The source assumes f holomorphic in the disk and continuous on its closure, with no zeros at the center or on the boundary. Its formulation follows by the same factorization: nonvanishing on the compact boundary and continuity give a zero-free collar, so the remaining zero set is finite; after factorization the logarithm of the nonvanishing factor is harmonic in the disk and continuous on its closure. Holomorphic extension past the circle is convenient for the boundary-zero statement above, but is not required for the source's nonvanishing-boundary case.

If f has a zero of order m at the center, write $f(z) = (z-a)^m g(z)$ with $g(a) \neq 0$. The correct replacement is

$$M_{\log|f|}(a,R) = m\log R + \log\left|\frac{f^{(m)}(a)}{m!}\right| + \sum_{0<|b_j-a|<R} \log\frac{R}{|b_j-a|}. \tag{8.1}$$

One must not use $\log|f(a)| = -\infty$ in a finite subtraction. Factoring the central zero removes that ambiguity and exposes its coefficient.

8.3 Counting functions and observable bounds

Fix a with $f(a) \neq 0$, and define

$$n_f(a,t) = \sum_{|b-a|<t} \operatorname{ord}_b(f).$$

On every compact disk this is a finite nondecreasing integer-valued step function. A zero exactly on a circle may be included or excluded without changing an integral in the radius variable. Termwise integration gives

$$\sum_{|b-a|<R} \operatorname{ord}_b(f) \log \frac{R}{|b-a|} = \int_0^R \frac{n_f(a,t)}{t}\,\mathrm{d}t.$$

Indeed a zero at distance $s > 0$ contributes $\int_s^R dt/t = \log(R/s)$, and the finite sum can be interchanged with the integral. Jensen therefore reads

$$M_{\log|f|}(a,R) - \log|f(a)| = \int_0^R \frac{n_f(a,t)}{t}\,\mathrm{d}t. \tag{8.2}$$

This is an integrated count. A small mean defect limits the number of zeros deep inside the circle, but does not give the same cost to zeros arbitrarily close to its boundary.

Corollary 8.4 (Interior zero-count bound)**.** *If $0 < r < R$ and $f(a) \neq 0$, then*

$$n_f(a,r) \log \frac{R}{r} \le M_{\log|f|}(a,R) - \log|f(a)| \le \log \frac{\max_{|z-a|=R}|f(z)|}{|f(a)|}.$$

Proof. Each zero with distance less than r contributes at least $\log(R/r)$ to Jensen's sum; all remaining contributions are nonnegative. For the second inequality use the pointwise upper bound $\log|f| \le \log \max_{|z-a|=R}|f(z)|$ on the circle and integrate. The denominator $\log(R/r)$ is positive, so division preserves the inequality. It tends to zero as $r \uparrow R$, correctly reflecting the loss of control over zeros near the observation circle. □

Writing $R = e^t$ makes the mechanism particularly transparent. For a polynomial $p(z) = c\prod_{j=1}^N (z - b_j)$ centered at zero,

$$M_{\log|p|}(0,e^t) = \log|c| + \sum_{j=1}^N \max\{t, \log|b_j|\},$$

where a zero at zero contributes t. The graph is convex and piecewise linear in t. Its slope jumps by the zero multiplicity at each radius $|b_j|$. This is an exact formula, not a finite-difference approximation for an unknown count.

8.4 Blaschke factors and three-circle comparison

For a zero $b \in \mathbb{D}$, put

$$B_b(z) = \frac{z-b}{1-\overline{b}z}, \qquad |B_b(e^{it})| = 1.$$

The equality follows by expanding both squared moduli: $|e^{it}-b|^2 = |1-\overline{b}e^{it}|^2$. If f is holomorphic on a neighborhood of $\overline{\mathbb{D}}$, with selected zeros $b_1, \dots, b_N$ listed no more often than their multiplicities, then $g = f/\prod_j B_{b_j}$ is holomorphic in $\mathbb{D}$ and continuous on its closure. Apparent poles at the selected zeros are removable by local factorization. On the boundary $|g| = |f|$, so the maximum modulus principle gives, with $M = \max_{|z|=1}|f(z)|$,

$$|f(z)| \le M \prod_{j=1}^N |B_{b_j}(z)| \quad (z \in \mathbb{D}).$$

At zero this is $|f(0)| \le M\prod_j |b_j|$. When $f(0) \neq 0$, taking logarithms recovers the corresponding finite Jensen upper bound, while the pointwise product retains the locations rather than only their radii.

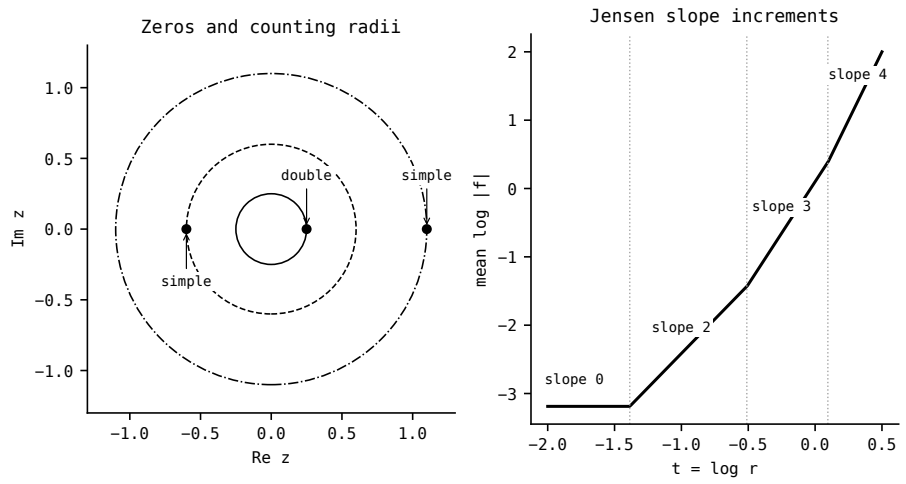


Figure 8.2: Jensen's mean for $f(z) = (z-1/4)^2(z+3/5)(z-11/10)$. The geometric panel records the zero locations and multiplicities. In logarithmic radius, the mean is the sum of four hinge functions; its slopes are $0, 2, 3, 4$ across the three zero radii. A double zero causes a jump of two.

Proposition 8.5 (Three-circle inequality)**.** *Let f be holomorphic on an annulus containing $\{r_1 \le |z| \le r_2\}$ and not identically zero. Set $M(r) = \max_{|z|=r} |f(z)|$. For $r_1 < r < r_2$,*

$$\log M(r) \le \frac{\log(r_2/r)}{\log(r_2/r_1)} \log M(r_1) + \frac{\log(r/r_1)}{\log(r_2/r_1)} \log M(r_2).$$

Proof. Neither boundary maximum is zero: otherwise the identity theorem would make f identically zero. There is a unique harmonic radial function $h(z) = A \log|z| + B$ with values $\log M(r_j)$ on the two circles. Solving the two linear equations for A, B gives the affine interpolation in $\log r$ appearing in the statement. Since $\log|f| \le h$ on both boundary circles and $\log|f|$ is subharmonic, comparison gives $\log|f(z)| \le h(z)$ throughout the annulus. Take the maximum on the intermediate circle. Zeros of f cause no difficulty because their logarithmic values are $-\infty$, which remain below the finite comparator. □

Thus both circular logarithmic means and logarithms of circular maxima have convexity in the logarithmic radius, but for different reasons. The mean has a mass-counting derivative; the maximum is controlled by annular harmonic comparison. They should not be substituted for one another in an equality.

Investigation 8.6 (A double zero and an exterior zero)**.** Compute $M_{\log|f|}(0, R)$ for $f(z) = (z - 1/4)^2(z - 2)$.

Solution. The translated-logarithm identity gives

$$M_{\log|f|}(0, R) = 2 \log \max(R, 1/4) + \log \max(R, 2).$$

At $R = 1$ this is $\log 2$. Since $|f(0)| = 1/8$, the Jensen defect is $\log 2 - \log(1/8) = \log 16 = 2 \log 4$, precisely the contribution of the double zero at $1/4$. The zero at 2 lies outside and contributes no defect, though it affects the additive mean itself.

Investigation 8.7 (A zero on the observation circle)**.** Evaluate $\frac{1}{2\pi} \int_0^{2\pi} \log|e^{it} - 1| \, \mathrm{d}t$ and explain why no infinite answer occurs.

Solution. The value is zero by Lemma 8.1. Near $t = 0$, $|e^{it} - 1| = 2|\sin(t/2)|$ is bounded above and below by positive constants times $|t|$. Therefore the only singularity has the integrable form

$\log|t| + O(1)$, and $\int_0^\delta |\log t|\,\mathrm{d}t = \delta(1-\log\delta) < \infty$. A pointwise value $-\infty$ on a set of angular measure zero need not make the integral infinite.

Investigation 8.8 (A certified analytic count bound)**.** Suppose f is holomorphic near $\overline{D(0,2)}$, $|f(0)| = 1$, and $|f(z)| \le 16$ on $|z| = 2$. How many zeros can lie in $|z| < 1/2$?

Solution. The bound gives $n_f(0,1/2)\log 4 \le \log 16 = 2\log 4$, hence at most two, counting multiplicity. This is a consequence of exact analytic hypotheses. If 16 were only a largest value observed on a finite sample, the boundary hypothesis would not have been established and the conclusion would not follow from that sample alone.

Investigation 8.9 (A central zero)**.** Let $f(z) = z^3(1 - z/2)$. Compute the mean on $|z| = 1$.

Solution. The factor z^3 contributes $3\log 1 = 0$, and $1 - z/2$ is nonvanishing on the disk with harmonic log modulus of center value zero. Thus the total mean is zero. In (8.1), $m = 3$ and $f^{(3)}(0)/3! = 1$; there are no other interior zeros. Using the unmodified expression $\log|f(0)|$ would instead create a meaningless finite equality with $-\infty$.

Investigation 8.10 (An infinite-disk consequence)**.** Let f be bounded and holomorphic on $\mathbb{D}$, $f(0) \neq 0$. Show that its zeros b_j, counted with multiplicity, satisfy $\sum_j (1 - |b_j|) < \infty$.

Solution. For $R < 1$ avoiding zero radii, Jensen on the compact subdisk gives

$$\sum_{|b_j|<R} \log\frac{R}{|b_j|} \le \log\frac{\|f\|_\infty}{|f(0)|}.$$

Each summand, set equal to zero for $R \le |b_j|$, increases with R. Monotone convergence for the counting measure on the zeros gives $\sum_j \log(1/|b_j|) < \infty$ as $R \uparrow 1$. Finally $1 - t \le -\log t$ for $0 < t \le 1$, by integrating $1/s \ge 1$ from t to 1. The asserted series is therefore bounded by the preceding one. This is a necessary zero-set condition, not a construction of a bounded function for every admissible sequence.

Investigation 8.11 (Sharpness of three circles)**.** For $f(z) = cz^m$ with $c \neq 0$ and an integer m, check equality in the three-circle inequality on any annulus avoiding zero.

Solution. One has $\log M(r) = \log|c| + m\log r$, an affine function of $\log r$. The interpolation is therefore exact. Negative m are allowed on an annulus but not for a holomorphic function on a disk containing zero. This distinguishes the annular statement from assumptions of holomorphic extension through the center.

Chapter 9

Logarithmic potentials and distributional sources

The zero-counting measure of a holomorphic function is discrete, with positive integer masses. Potential theory allows the masses to be arbitrary nonnegative real numbers, spread along curves, or distributed over area. The logarithmic kernel remains the same. What changes is the bookkeeping required to justify integration through its singularity.

9.1 The measure hypotheses and the sign convention

A positive Radon measure μ on an open subset of $\mathbb{C}$ is a countably additive nonnegative measure on its Borel sets, finite on compact subsets and regular with respect to open and compact approximation. The examples needed first are

$$\sum_{j=1}^{N} m_j \delta_{a_j}, \qquad \rho(z)\, dA(z), \qquad \sigma \text{ supported on a compact curve}, \qquad m_j \geq 0,\ \rho \geq 0.$$

Here $\delta_a(E)$ is one or zero according as $a \in E$ or $a \notin E$. A probability measure has total mass one. A finite measure has finite total mass, even if its support is not compact; the first potential theorem assumes both finiteness and compact support so that no condition at infinity is hidden.

For a finite positive compactly supported measure, define

$$P_\mu(z) = \int_{\mathbb{C}} \log|z - w| \, \mathrm{d}\mu(w). \tag{9.1}$$

For fixed z, the positive part of the integrand is bounded on the compact support. The negative part may have infinite integral, so $P_\mu(z)$ belongs to $[-\infty, \infty)$. Our convention has positive Laplacian:

$$\Delta P_\mu = 2\pi\mu.$$

The alternative kernel $\log(1/|z - w|)$ gives $-P_\mu$ and a negative Laplacian. Both conventions occur in potential theory; an energy or Green function formula must specify which it uses. No change of sign is implicit when moving between the chapters of this volume.

The measure-theoretic inputs are standard theorems, not new claims of the monograph. Tonelli permits interchange for a nonnegative measurable integrand with extended values. Fubini permits interchange for an absolutely integrable signed integrand. Dominated convergence requires one integrable bound independent of the limiting parameter. Each use below includes that bound or a reduction to a nonnegative integrand. The existence theorem representing a positive functional by a Radon measure will be stated explicitly when the Riesz measure is constructed in the next chapter.

9.2 A uniform integrability estimate for the logarithm

Lemma 9.1 (Area control of the kernel)**.** *If $K, S \subset \mathbb{C}$ are compact, then*

$$\sup_{w\in S}\int_K |\log|z-w||\,dA(z) < \infty.$$

Consequently $P_\mu \in L^1_{\mathrm{loc}}(\mathbb{C})$ for every finite positive compactly supported μ.

Proof. Choose $L > 1$ so large that $K \subset D(w, L)$ for all $w \in S$. Translation and polar integration give

$$\int_K |\log|z-w||\,dA(z) \le 2\pi\int_0^L r|\log r|\,\mathrm{d}r.$$

The part from zero to one equals $2\pi/4 = \pi/2$, since $\int_0^1 r(-\log r)\,dr = 1/4$. The part from one to L is

$$2\pi\left(\frac{L^2}{2}\log L - \frac{L^2}{4} + \frac{1}{4}\right),$$

a finite number independent of w. Tonelli applied to $|\log|z-w||$ yields

$$\int_K\int |\log|z-w||\,\mathrm{d}\mu(w)\,dA(z) \le \mu(\mathbb{C})\,2\pi\int_0^L r|\log r|\,\mathrm{d}r < \infty.$$

Thus the inner logarithmic integral is absolutely convergent for almost every $z \in K$, and its absolute value is integrable there. Exhausting $\mathbb{C}$ by compact disks proves local integrability. In particular the potential is not identically $-\infty$, even if it takes that value on some of the support. □

Theorem 9.2 (Positive logarithmic potentials)**.** *For every finite positive compactly supported measure μ, the potential P_μ is subharmonic on $\mathbb{C}$ and harmonic on $\mathbb{C}\setminus\operatorname{supp}\mu$. Its exact circular mean is*

$$M_{P_\mu}(a,r) = \int \log\max\{r,|a-w|\}\,\mathrm{d}\mu(w).$$

Proof. First prove upper semicontinuity. If $z_n \to z$, then all z_n and z lie in a compact set, so $\log|z_n - w| \le C$ for all $w \in \operatorname{supp}\mu$, with one finite C. Fatou's lemma applied to $C - \log|z_n - w| \ge 0$ gives

$$\limsup_n \int \log|z_n-w|\,\mathrm{d}\mu(w) \le \int \limsup_n \log|z_n-w|\,\mathrm{d}\mu(w) = P_\mu(z).$$

The last equality includes $w = z$, where both limits are $-\infty$.

For the circle formula choose C bounding $\log|a+re^{it}-w|$ above for all t and w in the compact support. Tonelli applies to $C - \log|a+re^{it}-w|$. The translated-logarithm identity evaluates its inner angular integral and gives

$$M_{P_\mu}(a,r) = \int \log\max\{r,|a-w|\}\,\mathrm{d}\mu(w).$$

The resulting integrand is bounded above and below on the support: its lower bound is $\log r$. Hence the circular mean is finite, and the same computation also verifies absolute integrability on that circle. Since $\log|a-w| \le \log\max\{r,|a-w|\}$, integration gives the submean inequality, including points with $P_\mu(a) = -\infty$. The preceding lemma supplies nontriviality.

On a compact set at positive distance δ from the support, every first and second derivative of $\log|z-w|$ is uniformly bounded in w. Differentiation under the finite measure is therefore allowed, and the kernel has Laplacian zero there. Thus P_μ is C^2 and harmonic off the support. Repeating the derivative argument gives smoothness there. □

For a measure of total mass m supported in $\overline{D(0,R_0)}$, the far-field estimate is explicit. If $|z| > 2R_0$, then

$$P_\mu(z) = m\log|z| + \int \log|1 - w/z|\,\mathrm{d}\mu(w), \qquad |P_\mu(z) - m\log|z|| \le \frac{2mR_0}{|z|}.$$

Indeed $|w/z| \le 1/2$ and $|\log|1-\zeta|| \le \sum_{n\ge1}|\zeta|^n/n \le 2|\zeta|$ on that disk. The coefficient of $\log|z|$ records total mass, while the error retains a quantitative support-radius dependence.

If z, z' each have distance at least $\delta > 0$ from the support, then

$$|P_\mu(z) - P_\mu(z')| \le \frac{m}{\delta}|z - z'|.$$

For each w, the reverse triangle inequality bounds the difference of $|z-w|$ and $|z'-w|$ by $|z-z'|$, and the derivative of $\log t$ on $[\delta,\infty)$ is at most $1/\delta$. Integrating proves the estimate without requiring the straight segment between z and z' to avoid the support. This is useful when two points lie on opposite sides of a curve.

9.3 Truncating the singularity with a quantified area error

For a finite positive compactly supported measure μ of mass m, define

$$P_{\mu,\varepsilon}(z) = \int \log\max\{\varepsilon, |z-w|\}\,\mathrm{d}\mu(w), \qquad \varepsilon > 0.$$

The truncated kernel is finite and has Lipschitz constant $1/\varepsilon$ as a function of its distance argument. The reverse triangle inequality therefore gives

$$|P_{\mu,\varepsilon}(z) - P_{\mu,\varepsilon}(z')| \le \frac{m}{\varepsilon}|z - z'|.$$

The bound is global; no path joining the two points has to avoid the support. For each z the truncated potentials decrease to $P_\mu(z)$ as $\varepsilon \downarrow 0$. To justify the limit even at a pole, fix one initial ε_0 and a common upper bound on the kernels for $0 < \varepsilon \le \varepsilon_0$, then apply monotone convergence to that bound minus the kernel. Thus the limit may be $-\infty$ without an undefined subtraction of two infinite integrals.

The excess of a single truncated kernel has an exact full-plane area:

$$\begin{aligned}\int_{\mathbb{C}} (\log\max\{\varepsilon,|z-w|\} - \log|z-w|)\,dA(z) &= 2\pi\int_0^\varepsilon r\log(\varepsilon/r)\,\mathrm{d}r \\ &= 2\pi\varepsilon^2\int_0^1 s(-\log s)\,\mathrm{d}s = \frac{\pi\varepsilon^2}{2}.\end{aligned}$$

Tonelli applies because this excess is nonnegative. It follows, first almost everywhere and then as an L^1 statement, that

$$0 \le \int_K (P_{\mu,\varepsilon} - P_\mu)\,dA \le \frac{\pi m\varepsilon^2}{2} \quad \text{for every measurable } K \subset \mathbb{C}.$$

On the full plane equality holds. Neither individual potential need be integrable on the full plane; their nonnegative difference is integrable, and only that difference occurs in the displayed assertion. This distinction is necessary because each potential normally grows like $m\log|z|$.

The circular mean identity identifies the approximation geometrically:

$$P_{\mu,\varepsilon}(z) = \frac{1}{2\pi}\int_0^{2\pi} P_\mu(z + \varepsilon e^{it})\,\mathrm{d}t.$$

Equivalently it is the logarithmic potential of the measure obtained by replacing each source point by uniform probability on its radius-ε circle. Tonelli, applied after subtracting a common upper bound, proves this representation. In particular the truncated potential is subharmonic. It is generally only Lipschitz, and should not be confused with the smooth compact-kernel regularization proved in the next chapter. The area estimate here controls the singular truncation directly, with a constant independent of the arrangement of the source points.

A separate exterior approximation records angular information. If the support lies in $|w| \le R_0$, put $M_n = \int w^n \, \mathrm{d}\mu(w)$ and $q = R_0/|z| < 1$. Uniform convergence of the logarithm series on the support gives

$$P_\mu(z) = m \log |z| - \operatorname{Re} \sum_{n=1}^{N} \frac{M_n}{n z^n} + E_N(z), \qquad |E_N(z)| \le \frac{m q^{N+1}}{(N+1)(1-q)}.$$

The estimate uses $|M_n| \le m R_0^n$ and $\sum_{n>N} q^n/n \le q^{N+1}/[(N+1)(1-q)]$. It is uniform on every region $|z| \ge R > R_0$. Rotationally invariant measures have $M_n = 0$ for $n \ge 1$, so all such measures with the same mass have the same exterior potential. The circle and disk examples therefore express a real nonuniqueness of the interior source from exterior observations alone, not a failure to calculate enough terms.

9.4 The distributional identity and its normalization

A test function is a real smooth function $\varphi \in C_c^\infty(\Omega)$ with compact support in Ω. For $v \in L^1_{\mathrm{loc}}(\Omega)$, its distributional Laplacian is the linear functional

$$\langle \Delta v, \varphi \rangle = \int_\Omega v \Delta \varphi \, dA.$$

There is no minus sign: two distributional derivatives cancel their two minus signs. If v is C^2, integration by parts twice gives $\langle \Delta v, \varphi \rangle = \int (\Delta v) \varphi \, dA$ because the test function vanishes near the outer boundary. The definition therefore extends the classical Laplacian without changing its convention.

Theorem 9.3 (One point source)**.** *In distributions on* $\mathbb{C}$*,*

$$\Delta \log |z - a| = 2\pi \delta_a.$$

Proof. Translate to $a = 0$, and choose R with $\operatorname{supp} \varphi \subset D(0, R)$ and $\varphi = 0$ near $|z| = R$. On the punctured annulus $A_{\varepsilon,R}$ the logarithm is harmonic. Green's second identity gives

$$\int_{A_{\varepsilon,R}} \log |z| \Delta \varphi \, dA = \int_{\partial A_{\varepsilon,R}} (\log |z| \, \partial_n \varphi - \varphi \, \partial_n \log |z|) \, ds.$$

The outer boundary term vanishes. On the inner circle the outward normal for the annulus is $-\partial_r$, so $\partial_n \log |z| = -1/\varepsilon$ and $ds = \varepsilon \, dt$. The remaining expression is

$$-\varepsilon \log \varepsilon \int_0^{2\pi} \partial_r \varphi(\varepsilon e^{it}) \, \mathrm{d}t + \int_0^{2\pi} \varphi(\varepsilon e^{it}) \, \mathrm{d}t.$$

The first term tends to zero because $\nabla \varphi$ is bounded and $\varepsilon |\log \varepsilon| \to 0$; the second tends to $2\pi \varphi(0)$ by continuity. Finally the integral omitted over $D(0, \varepsilon)$ is bounded in absolute value by

$$2\pi \|\Delta \varphi\|_\infty \int_0^\varepsilon r |\log r| \, \mathrm{d}r \longrightarrow 0.$$

For $0 < \varepsilon < 1$, these three errors admit one explicit estimate:

$$\left| \int_{\mathbb{C}} \log|z| \Delta\varphi \, dA - 2\pi\varphi(0) \right| \le 2\pi\varepsilon |\log\varepsilon| \|\nabla\varphi\|_\infty + 2\pi\varepsilon \|\nabla\varphi\|_\infty + \pi\varepsilon^2 (1/2 - \log\varepsilon) \|\Delta\varphi\|_\infty.$$

The second term uses $|\varphi(\varepsilon e^{it}) - \varphi(0)| \le \varepsilon \|\nabla\varphi\|_\infty$ along the radial segment. For the last term, $\int_0^\varepsilon r(-\log r) dr = \varepsilon^2(1/4 - \frac{1}{2}\log\varepsilon)$. Every norm refers to the fixed test function, so the constants do not change while the puncture shrinks. The left side is independent of ε; a bound tending to zero therefore forces exact equality. In particular the factor 2π is a boundary flux, not a choice made by renormalizing the final formula. The argument also shows why replacing $\log r$ by $-\log r$ reverses the source sign.

Passing to the limit gives $\int \log|z| \Delta\varphi \, dA = 2\pi\varphi(0)$, as required. The inward orientation of the puncture boundary is what determines the positive sign. □

Corollary 9.4 (Laplacian of a potential)**.** *For finite positive compactly supported μ,*

$$\langle \Delta P_\mu, \varphi \rangle = 2\pi \int \varphi \, \mathrm{d}\mu.$$

Proof. The kernel integrability estimate, multiplied by $\|\Delta\varphi\|_\infty$, proves absolute integrability of the double integral over $\operatorname{supp}\varphi \times \operatorname{supp}\mu$. Fubini therefore gives

$$\int P_\mu(z) \Delta\varphi(z) \, dA(z) = \int \left[\int \log|z - w| \Delta\varphi(z) \, dA(z) \right] \mathrm{d}\mu(w) = 2\pi \int \varphi(w) \, \mathrm{d}\mu(w).$$

This is an identity for every test function, not a statement restricted to points outside the support. □

Corollary 9.5 (Zeros as a distributional measure)**.** *For nonzero holomorphic f on Ω,*

$$\frac{1}{2\pi} \Delta \log|f| = \sum_{f(b)=0} \operatorname{ord}_b(f) \, \delta_b.$$

Proof. The zeros are locally finite. Near each zero factor $f(z) = (z - b)^m g(z)$ with g nonzero. The harmonic term $\log|g|$ has zero distributional Laplacian, and the logarithmic term contributes $2\pi m \delta_b$. Away from the zeros the whole logarithm is harmonic. A compactly supported test function meets only finitely many zeros, so these local identities combine using a smooth partition of unity on its compact support. Equivalently, factor all zeros in a small neighborhood of that support and apply linearity. No sum over infinitely many zeros is interchanged on a fixed compact test region. □

9.5 Two exact continuous charge distributions

Let σ be uniform probability measure on the unit circle: $\int g \, \mathrm{d}\sigma = (2\pi)^{-1} \int_0^{2\pi} g(e^{it}) \, \mathrm{d}t$. The translated-logarithm formula gives the entire potential explicitly:

$$P_\sigma(z) = \log^+|z|.$$

It is constant inside, logarithmic outside, and continuous across the circle. Its distributional Laplacian is $2\pi\sigma$, concentrated on the seam. The radial derivative is zero on the inner side and one on the outer side of $r = 1$; this positive jump is consistent with the positive circular source.

Now let $\mu = \pi^{-1}\mathbf{1}_{\mathbb{D}}\, dA$, normalized area measure on the unit disk. Rotational symmetry and the logarithmic mean identity give, for $r = |z|$,

$$P_\mu(r) = 2\int_0^1 s \log\max\{r, s\}\, \mathrm{d}s = \begin{cases} (r^2-1)/2, & 0 \le r \le 1, \\ \log r, & r \ge 1. \end{cases}$$

For $0 < r < 1$, split the integral at $s = r$:

$$2\int_0^r s\log r\, \mathrm{d}s + 2\int_r^1 s\log s\, \mathrm{d}s = r^2\log r + \left[s^2\log s - s^2/2\right]_r^1 = \frac{r^2-1}{2}.$$

At $r = 0$ the same expression follows by evaluating $2\int_0^1 s\log s\, ds = -1/2$. The first derivative agrees at $r = 1$, so there is no additional circle mass. Inside the disk the Laplacian is 2, exactly 2π times the area density $1/\pi$; outside it is zero. Both distributions have total mass one and identical exterior potential, but different interior potentials and different source supports.

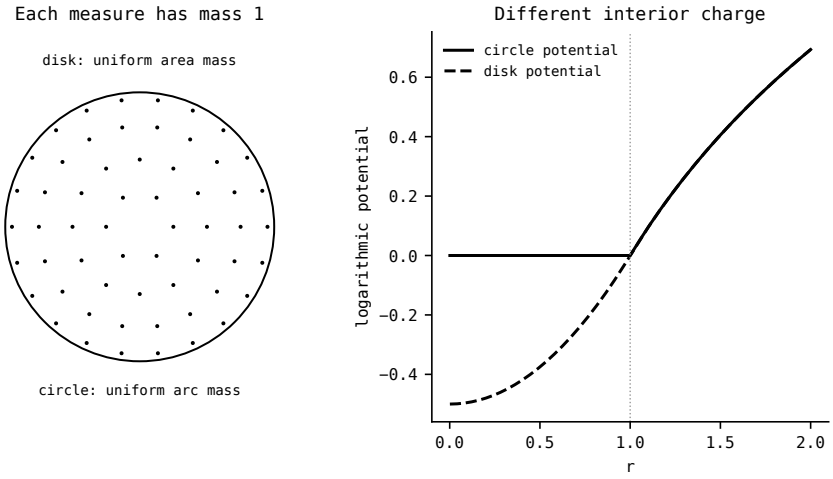


Figure 9.1: Two probability measures with the same exterior logarithmic field. Uniform circle charge produces $\log^+ r$; uniform disk charge produces $(r^2-1)/2$ inside and $\log r$ outside. The radial graphs and support diagrams distinguish a derivative jump on the circle from a continuous derivative with area density in the disk.

Investigation 9.6 (A finite weighted potential)**.** For $\mu = 2\delta_{-1} + 3\delta_1$, compute its potential, Laplacian, and leading behavior at infinity.

Solution. The potential is $P_\mu(z) = 2\log|z+1| + 3\log|z-1|$, and $\Delta P_\mu = 4\pi\delta_{-1} + 6\pi\delta_1$. Its total mass is five, so $P_\mu(z) = 5\log|z| + O(1/|z|)$. For $|z| > 2$ the explicit preceding estimate gives $|P_\mu(z) - 5\log|z|| \le 10/|z|$. Each pole retains its own coefficient even though the leading far field sees only their sum.

Investigation 9.7 (Scaling a uniform disk)**.** Find the potential of probability area measure on $D(0, R)$.

Solution. Under $w = R\zeta$, the measure is the pushforward of $\pi^{-1}\mathbf{1}_{\mathbb{D}}dA$. Hence

$$P_R(z) = \log R + P_1(z/R) = \begin{cases} \log R + (|z|^2/R^2 - 1)/2, & |z| \le R, \\ \log|z|, & |z| \ge R. \end{cases}$$

The interior Laplacian is $2/R^2$, which equals 2π times the density $1/(\pi R^2)$. The expression at the origin is finite: $P_R(0) = \log R - 1/2$.

Investigation 9.8 (Equal exterior fields do not determine the interior)**.** Can the potential outside the unit disk distinguish uniform circle probability from uniform disk probability?

Solution. No: both equal $\log|z|$ for $|z|>1$. At zero, however, the circle potential is 0 and the disk potential is $-1/2$. Their measures are therefore different. Distributional equality on all of $\mathbb{C}$ would determine the measure via $\Delta P = 2\pi\mu$, but equality only on an exterior open set determines neither the interior Laplacian nor the interior potential.

Investigation 9.9 (The normalization test)**.** What is the distributional Laplacian of $u(z) = -(2\pi)^{-1}\log|z-a|$?

Solution. It is $-\delta_a$. Thus this function is a fundamental solution for $-\Delta$, while $(2\pi)^{-1}\log|z-a|$ is a fundamental solution for Δ. A formula with a positive Green function for $-\Delta$ should not be combined with the positive-source convention for Δ without changing the sign.

Investigation 9.10 (Boundary mass of a radial corner)**.** For $a>0$ determine the measure associated with $u(z) = a\log^+|z|$.

Solution. By the circle-potential identity, $u = P_{a\sigma}$ and $(2\pi)^{-1}\Delta u = a\sigma$. In arclength notation on the unit circle, $\sigma = ds/(2\pi)$, so $\Delta u = a\,ds$ there. The total Laplacian mass is $2\pi a$. Confusing angular probability with unnormalized arclength would lose the factor 2π.

Investigation 9.11 (Why compact support was specified)**.** Give a positive finite measure for which the logarithmic potential has an infinite positive part.

Solution. Put $\mu = \sum_{n\geq 1} n^{-2}\delta_{e^n}$. The total mass $\sum n^{-2}$ is finite, but at zero

$$\int \log|w|\,\mathrm{d}\mu(w) = \sum_{n\geq 1}\frac{n}{n^2} = \infty.$$

Thus the potential would take $+\infty$, outside our subharmonic value convention. Compact support can be replaced by an appropriate logarithmic moment condition, but finite mass alone does not imply that condition.

Chapter 10

Regularization, Riesz measures, and the recovery of a function

The logarithmic potential theorem starts with a measure and constructs a subharmonic function. The converse asks whether every subharmonic function contains a positive source measure of this kind. Its local answer is yes, up to a harmonic term. To make that statement useful, three different objects must be distinguished: the pointwise upper semicontinuous function, its locally integrable equivalence class, and the measure represented by its distributional Laplacian. The previous chapters supplied the bridge between point values and almost-everywhere values. Smooth approximation will supply the bridge to the Laplacian.

10.1 A positive smooth approximate identity

Choose a nonnegative radial $\rho \in C_c^\infty(\mathbb{C})$ with $\operatorname{supp}\rho \subset \overline{\mathbb{D}}$ and $\int \rho\, dA = 1$. For example, normalize

$$\rho(z) = c\begin{cases}\exp(-1/(1-|z|^2)), & |z|<1,\\ 0, & |z|\geq 1,\end{cases}$$

where $c>0$ is the reciprocal of its positive finite area integral. Put $\rho_\varepsilon(z) = \varepsilon^{-2}\rho(z/\varepsilon)$. For a subharmonic u on Ω, define

$$u_\varepsilon(z) = \int u(z-w)\rho_\varepsilon(w)\, dA(w), \qquad \operatorname{dist}(z, \mathbb{C}\setminus\Omega) > \varepsilon.$$

Local integrability makes the convolution finite. Differentiating the smooth compactly supported kernel shows that u_ε is smooth; for a multi-index α,

$$D^\alpha u_\varepsilon(z) = \int u(w) D^\alpha \rho_\varepsilon(z-w)\, dA(w).$$

On each compact subdomain the derivative kernel is bounded and supported in a fixed compact set, so local integrability justifies the derivatives. The support restriction is important: no values outside Ω are assigned by this construction.

Theorem 10.1 (Smooth subharmonic approximation)**.** *For all sufficiently small ε on any fixed relatively compact subdomain, u_ε is smooth subharmonic. Moreover*

$$u_\varepsilon(a) \geq u(a), \qquad u_\varepsilon(a) \longrightarrow u(a) \quad (\varepsilon \downarrow 0), \qquad u_\varepsilon \to u \text{ in } L^1_{\mathrm{loc}}.$$

Proof. To verify the submean inequality, choose a circle with its closed ε-neighborhood in Ω. Applying the submean inequality at every translated center and integrating against the

nonnegative kernel gives

$$\begin{aligned} u_\varepsilon(a) &\le \int \rho_\varepsilon(w) \left[\frac{1}{2\pi} \int_0^{2\pi} u(a - w + re^{it})\, \mathrm{d}t \right] dA(w) \\ &= \frac{1}{2\pi} \int_0^{2\pi} u_\varepsilon(a + re^{it})\, \mathrm{d}t. \end{aligned}$$

The interchange is justified by absolute integrability. Indeed $\rho_\varepsilon \le \|\rho_\varepsilon\|_\infty$, and all translated integration disks lie in one compact set $K \subset \Omega$, so the absolute double integral is bounded by $2\pi\|\rho_\varepsilon\|_\infty \int_K |u|\, dA$.

Write the radial kernel as $\rho_\varepsilon(w) = q_\varepsilon(|w|)$. Polar integration expresses the convolution as a weighted average of circular means:

$$u_\varepsilon(a) = \int_0^\varepsilon 2\pi r q_\varepsilon(r) M_u(a, r)\, \mathrm{d}r, \qquad \int_0^\varepsilon 2\pi r q_\varepsilon(r)\, \mathrm{d}r = 1.$$

Each mean is at least $u(a)$, proving the lower bound. Upper semicontinuity gives the upper bound $u_\varepsilon(a) \le u(a) + \eta$ for small ε when $u(a)$ is finite. If $u(a) = -\infty$, the same argument gives $u_\varepsilon(a) \le B$ for every prescribed real B once ε is small. This proves pointwise convergence at all points, not merely almost everywhere.

Differentiation of the convolution can be quantified separately from its pointwise limit. With $C_\alpha = \|D^\alpha \rho\|_\infty$ and $K_\varepsilon = K + \overline{D(0,\varepsilon)} \Subset \Omega$,

$$\sup_{z \in K} |D^\alpha u_\varepsilon(z)| \le C_\alpha \varepsilon^{-2-|\alpha|} \|u\|_{L^1(K_\varepsilon)}.$$

The power is obtained directly from $D^\alpha \rho_\varepsilon(w) = \varepsilon^{-2-|\alpha|}(D^\alpha \rho)(w/\varepsilon)$. For fixed positive ε, difference quotients of the kernel are uniformly bounded on a common compact support, so dominated convergence justifies each derivative. The constants generally blow up as $\varepsilon \downarrow 0$. Smoothness of each approximation does not therefore assert smoothness of the limit. A logarithmic pole is exactly the kind of limit for which such a conclusion would be false.

For the L^1 assertion, first multiply u by a smooth cutoff equal to one on a neighborhood of the chosen compact set, obtaining an $L^1(\mathbb{C})$ function v which agrees with u wherever the small convolutions are used. Translations are continuous in L^1: $\|v(\cdot - w) - v\|_1 \to 0$ as $w \to 0$. One proof approximates v in L^1 by a continuous compactly supported function, for which uniform continuity and a common compact support prove the assertion; the two approximation errors are unchanged under translation. Density of such continuous functions is the standard Lebesgue approximation theorem. Then

$$\|v * \rho_\varepsilon - v\|_1 \le \int \rho_\varepsilon(w) \|v(\cdot - w) - v\|_1\, dA(w) \le \sup_{|w| \le \varepsilon} \|v(\cdot - w) - v\|_1 \to 0.$$

Here is the epsilon argument behind the translation limit, including the uniformity in $|w| \le \varepsilon$. Given $\eta > 0$, choose $g \in C_c(\mathbb{C})$ with $\|v - g\|_1 < \eta$. Translation preserves Lebesgue area and hence the L^1 norm, so

$$\|v(\cdot - w) - v\|_1 \le 2\eta + \|g(\cdot - w) - g\|_1.$$

For $|w| \le 1$, both terms in the last difference have support in one fixed compact set E, of finite area. If $\omega_g(\delta)$ is the modulus of uniform continuity of g, then

$$\sup_{|w| \le \varepsilon} \|v(\cdot - w) - v\|_1 \le 2\eta + |E|\omega_g(\varepsilon).$$

First choose ε small, and then let η tend to zero. The approximation g is fixed during the first limit. This order avoids assuming a common continuity modulus for all possible L^1 approximants. It also proves convergence for every sufficiently small smoothing radius, rather than only along a specially selected sequence.

Restriction to the selected compact set completes the proof. □

The theorem does not identify the smoothing with a physical experiment. The parameter ε specifies an analytic convolution and a limit; no sampled graph proves convergence. It also does not allow arbitrary sign-changing kernels: positivity was used in both the mean inequality and the lower pointwise bound.

Corollary 10.2 (Positive distributional Laplacian)**.** *For subharmonic u and every nonnegative test function φ,*

$$\langle \Delta u, \varphi \rangle = \int u \Delta\varphi \, dA \geq 0.$$

Proof. On a neighborhood of $\operatorname{supp}\varphi$, the smooth criterion gives $\Delta u_\varepsilon \geq 0$. Therefore

$$\int u_\varepsilon \Delta\varphi \, dA = \int \varphi \Delta u_\varepsilon \, dA \geq 0.$$

The left side tends to $\int u\Delta\varphi$ by local L^1 convergence and boundedness of $\Delta\varphi$. Taking the limit preserves the inequality. □

10.2 From a positive functional to a measure

We use one standard representation theorem from measure theory: *a positive linear functional on continuous compactly supported real functions on a locally compact second-countable metric space is integration against a unique positive Radon measure, provided it is bounded on functions supported in each fixed compact set.* This is the positive-functional Riesz representation theorem. It is a measure-theoretic input here; it is distinct from the subharmonic Riesz decomposition, which will be proved below. Positivity itself supplies the needed local boundedness.

Theorem 10.3 (The Riesz measure)**.** *For every subharmonic u on Ω there is a unique positive Radon measure μ_u such that*

$$\int \varphi \, \mathrm{d}\mu_u = \frac{1}{2\pi} \int u \Delta\varphi \, dA \quad (\varphi \in C_c^\infty(\Omega)).$$

Thus $\Delta u = 2\pi\mu_u$ in distributions.

Proof. Define $T(\varphi) = (2\pi)^{-1} \int u\Delta\varphi$. This is a positive linear functional on real test functions by the preceding corollary. Fix a compact $K \subset \Omega$ and choose a nonnegative smooth cutoff χ equal to one on a neighborhood of K and supported in Ω. For a real test function supported in K, put $M = \|\varphi\|_\infty$. The inequalities $-M\chi \leq \varphi \leq M\chi$ and positivity imply

$$|T(\varphi)| \leq MT(\chi).$$

Hence on each fixed compact support T is bounded in the uniform norm, although its original definition used two derivatives.

For $f \in C_c(\Omega)$, approximate f uniformly by smooth compactly supported functions with support in one fixed compact neighborhood of $\operatorname{supp} f$. Extend f by zero and convolve with a smooth compact kernel; uniform continuity gives this approximation. The displayed bound makes T of the approximants a Cauchy sequence and makes its limit independent of the choice of approximation. Define $T(f)$ to be that limit. If $f \geq 0$, positive mollifiers give nonnegative approximants, so the extension remains positive. It is linear and locally uniformly bounded. More explicitly, let f_j, g_j be two such smooth approximation sequences, with all supports contained in one compact neighborhood $K' \Subset \Omega$. Choose $\chi' = 1$ near K'. Then

$$|T(f_j) - T(g_k)| \leq T(\chi')\|f_j - g_k\|_\infty \leq T(\chi')(\|f_j - f\|_\infty + \|g_k - f\|_\infty).$$

Both terms tend to zero independently, proving that the extension has one value for f, even when the two sequences converge at different rates. For several continuous functions, take the union of their compact supports and one common cutoff; applying the same estimate to linear combinations proves linearity of the extension. Positivity is retained by choosing the mollified approximants nonnegative, rather than merely assuming that a uniformly small approximation error has the right sign.

The local functional bound is finite for a concrete reason:

$$0 \le T(\chi') \le \frac{\|\Delta\chi'\|_\infty}{2\pi} \int_{\operatorname{supp}\chi'} |u|\, dA < \infty.$$

Once the representing measure exists, $\chi' \ge 1$ on K' gives $\mu_u(K') \le \int \chi'\, d\mu_u = T(\chi')$. Thus the very estimate used to construct the functional controls local source mass. This argument checks the hypotheses of the named measure representation input; it does not purport to replace that representation theorem with the formal notation Δu.

The stated representation theorem now gives a unique Radon measure μ_u representing this extension. In particular it represents the original functional on test functions. Conversely, equality on all smooth compactly supported functions implies equality on all continuous compactly supported functions by the same uniform approximation, so uniqueness in the representation theorem gives the required uniqueness. □

For smooth u, this measure is simply $\mu_u = (2\pi)^{-1}(\Delta u)dA$. For a logarithmic potential it is the original measure μ. For $\log|f|$ it is the zero-counting measure, including multiplicities. These are consequences of the same defining identity, so their constants can be checked against one another.

10.3 A distributionally harmonic function has a harmonic representative

Lemma 10.4 (Weyl's lemma in the locally integrable case)**.** *If $v \in L^1_{\mathrm{loc}}(\Omega)$ and $\int v\Delta\varphi\, dA = 0$ for every test function, there is a harmonic h on Ω with $h = v$ almost everywhere.*

Proof. Convolve v with the compact smooth kernel just used. At points whose ε-neighborhood lies in Ω,

$$\Delta v_\varepsilon(z) = \int v(w)\Delta_z\rho_\varepsilon(z-w)\, dA(w) = 0,$$

because $w \mapsto \rho_\varepsilon(z-w)$ is a test function and its z-Laplacian equals its w-Laplacian. Thus v_ε is harmonic. Fix $K \Subset \Omega$ and choose $r > 0$ so that every disk $D(z, 2r)$ with $z \in K$ lies in a larger relatively compact open set W. For small ε, the harmonic area mean and the triangle inequality give

$$|v_\varepsilon(z)| \le \frac{1}{\pi r^2} \int_{D(z,r)} |v_\varepsilon|\, dA \le \frac{1}{\pi r^2} \int_W |v|\, dA.$$

For the last inequality, Tonelli and the positive convolution kernel show that the integral on the smaller disk is at most the integral of $|v|$ over its ε-neighborhood, contained in W. This estimate also gives direct control of differences. If $K + \overline{D(0,2r)} \subset W \Subset \Omega$ and ε, η are small enough that both convolutions are harmonic on W, the same area argument applied to their harmonic difference yields

$$\sup_K |v_\varepsilon - v_\eta| \le \frac{1}{\pi r^2}\|v_\varepsilon - v_\eta\|_{L^1(W)} \le \frac{\|v_\varepsilon - v\|_{L^1(W)} + \|v_\eta - v\|_{L^1(W)}}{\pi r^2}.$$

Both terms tend to zero by the approximate-identity theorem, so the full family, not only a chosen subsequence, is uniformly Cauchy on K. The separation of scales matters: the right side is measured on W, while the conclusion holds on the smaller compact set. There is no claim that an L^1 bound controls arbitrary nonsmooth functions pointwise.

The approximants are therefore locally uniformly bounded harmonic functions. Harmonic normality from Proposition 4.5 supplies, by a compact exhaustion and diagonal selection, a subsequence converging locally uniformly to a harmonic h. The approximate-identity argument also gives $v_\varepsilon \to v$ in L^1_{loc}. On each compact set, both limits must agree almost everywhere, so $h = v$ almost everywhere. If two harmonic representatives existed, their continuous difference would vanish almost everywhere and hence everywhere. This also establishes uniqueness of the representative. □

Only the locally integrable version is needed here. It has been proved using harmonic normality and convolution; no unmentioned elliptic regularity theorem is required at this point.

The representative and its first derivatives. After the representative h has been identified, the difference estimate can be sharpened to

$$\sup_K |v_\varepsilon - h| \le \frac{1}{\pi r^2}\|v_\varepsilon - v\|_{L^1(W)}$$

whenever every radius-r disk about K is contained in W and the approximant is harmonic there. The equality $h = v$ almost everywhere is used on the right, where changing a representative does not change the norm. Thus the bound proves convergence of the whole approximate family and uniqueness of its pointwise limit simultaneously.

There is also a concrete first-derivative bound. For a harmonic function f on a neighborhood of $\overline{D(z,2r)}$, the area mean gives

$$\sup_{|w-z|=r} |f(w)| \le \frac{1}{\pi r^2}\int_{D(z,2r)} |f|\,dA.$$

Differentiate the disk Poisson formula at its center in a unit direction $e = e^{i\alpha}$:

$$\partial_e f(z) = \frac{1}{\pi r}\int_0^{2\pi} f(z + re^{it})\cos(t-\alpha)\,\mathrm{d}t.$$

Since $\int_0^{2\pi} |\cos(t-\alpha)|dt = 4$, taking the supremum over unit directions yields

$$|\nabla f(z)| \le \frac{4}{\pi^2 r^3}\|f\|_{L^1(D(z,2r))}.$$

Apply this to $f = v_\varepsilon - h$ on a slightly smaller compact set. The already proved L^1 convergence then gives locally uniform convergence of first derivatives as well. This is a derived interior estimate for harmonic functions, not an invocation of general elliptic regularity. It explains why the harmonic remainder in the Riesz decomposition has classical derivatives even though its initial construction was only as an equivalence class of integrable functions.

Theorem 10.5 (Local Riesz decomposition)**.** *Let u be subharmonic on Ω and let $\overline{D(a,R)} \subset \Omega$. Put $\nu = \mu_u|_{D(a,R)}$, the restriction of its Riesz measure to the open disk. Then there is a harmonic function h on $D(a,R)$ such that*

$$u(z) = h(z) + \int_{D(a,R)} \log|z-w|\,\mathrm{d}\mu_u(w) \quad (z \in D(a,R)).$$

Equality holds at every point, with $-\infty$ allowed on both sides.

Proof. The restricted measure is finite because $\mu_u(D(a,R)) \le \mu_u(\overline{D(a,R)}) < \infty$, and its support as a measure on $\mathbb{C}$ is contained in the compact closed disk. Its potential P_ν is therefore locally integrable and subharmonic. The difference $u - P_\nu$ is defined almost everywhere as an L^1 function. At a point where both terms are $-\infty$, no pointwise subtraction is made. For every test function supported in $D(a,R)$,

$$\Delta(u - P_\nu) = 2\pi\mu_u - 2\pi\nu = 0$$

in distributions. Weyl's lemma gives a harmonic h with $u - P_\nu = h$ almost everywhere in the disk. Both u and $P_\nu + h$ are subharmonic. They agree almost everywhere, so the canonical-value identity in Theorem 6.6 makes them equal everywhere. This final step restores all singular point values without subtracting infinities. □

The restoration of values can be written as an actual limit. For each small disk centered at z in the localization disk, almost-everywhere equality gives

$$A_t u(z) = A_t P_\nu(z) + A_t h(z) = A_t P_\nu(z) + h(z).$$

Each average is finite by local integrability. Sending $t \downarrow 0$ yields $u(z) = P_\nu(z) + h(z)$ by canonical values. If the potential tends to $-\infty$ in this limit, adding the fixed finite number $h(z)$ leaves that limit $-\infty$. Consequently no step subtracts $-\infty$ from $-\infty$; the subtraction occurred only between L^1 representatives before the pointwise identity was restored.

For nested localization disks $D_1 = D(a,R_1) \Subset D_2 = D(a,R_2)$, write $\nu_j = \mu_u|_{D_j}$ and $u = h_j + P_{\nu_j}$. The difference measure $\lambda = \nu_2 - \nu_1$ is positive, finite, and supported on $\overline{D_2} \setminus D_1$. Its potential is finite and harmonic on D_1. Additivity of the integral gives $P_{\nu_2} = P_{\nu_1} + P_\lambda$, with no ambiguous sum because P_λ is finite there. At almost every point,

$$h_1 - h_2 = P_\lambda.$$

Both sides are harmonic on D_1, so their continuous difference vanishes everywhere, not merely almost everywhere. On a compact subset $K \Subset D_1$, put $\delta = \operatorname{dist}(K, \operatorname{supp}\lambda) > 0$ if $\lambda \neq 0$. Differentiation under the finite measure gives

$$|\nabla(h_1 - h_2)(z)| \le \frac{\lambda(\mathbb{C})}{\delta} \quad (z \in K).$$

The added exterior sources thus modify the harmonic remainder in a controlled way. If $\lambda = 0$, the remainders agree and the bound is unnecessary. Sources on the intermediate circle belong to λ because ν_1 was restricted to an open disk; they remain a positive distance from every such K.

The harmonic term depends on the chosen localization disk. The measure μ_u does not. Enlarging the disk adds a potential whose sources lie outside the smaller disk; that added potential is harmonic there and is compensated by the change in h. This explains why local representations can be consistent without one fixed global harmonic remainder on an arbitrary domain.

Corollary 10.6 (Distributional characterization)**.** *For $v \in L^1_{\mathrm{loc}}(\Omega)$, the condition $\int v\Delta\varphi \ge 0$ for every nonnegative test function is equivalent to the existence of a unique subharmonic representative $u = v$ almost everywhere, component by component.*

Proof. The forward implication in the direction from subharmonicity to a positive distribution was proved above. Conversely use the same positive-functional argument to represent $(2\pi)^{-1}\Delta v$ by a positive Radon measure μ. On a relatively compact disk restrict μ to that disk and form P_ν. Then $v - P_\nu$ has zero distributional Laplacian, hence is almost everywhere a harmonic function h. The function $P_\nu + h$ is a subharmonic representative on the disk. On overlaps, two such representatives agree almost everywhere because both represent v. The canonical-value theorem makes them equal everywhere, so they glue to a subharmonic function on Ω. Nontriviality on each component follows from local integrability, and uniqueness follows again from canonical values. □

10.4 General Jensen formula and radial mass

Theorem 10.7 (Jensen–Riesz identity)**.** *Let u be subharmonic on a neighborhood of $\overline{D(a,R)}$ and suppose $u(a) > -\infty$. Then for $0 < r \le R$,*

$$M_u(a,r) - u(a) = \int_{|w-a|<r} \log \frac{r}{|w-a|}\, \mathrm{d}\mu_u(w) = \int_0^r \frac{\mu_u(D(a,t))}{t}\, \mathrm{d}t.$$

Both sides are finite. If the center is a pole, no subtraction of $u(a) = -\infty$ is asserted.

Proof. Apply the local Riesz decomposition on a slightly larger concentric disk. The harmonic term has mean equal to its center value. The potential mean formula gives

$$M_{P_\nu}(a,r) = \int \log \max\{r, |a-w|\}\, \mathrm{d}\nu(w).$$

Since $u(a)$ and the harmonic term are finite, $P_\nu(a)$ is finite. For completeness, the potential at the center is an absolutely convergent integral in this finite-center case. Its positive part is bounded by a constant times $\nu(\mathbb{C})$ because the source set is compact. A finite value of $P_\nu(a)$ therefore forces its negative part to be finite too. For a fixed positive r, the mean kernel $\log \max\{r, |a-w|\}$ lies between $\log r$ and one finite upper bound, so it is absolutely integrable against ν. These are the two finite integrals being subtracted. An atom at a would make the first one $-\infty$, so the finite-center hypothesis also implies $\mu_u(\{a\}) = 0$. It does not require that a whole neighborhood of a be free of source mass.

The two integrals can therefore be subtracted; their difference is zero for $|a-w| \ge r$ and $\log(r/|a-w|)$ inside. This proves the first identity and its finiteness. For the second write the nonnegative kernel as

$$\log \frac{r}{|a-w|}\, \mathbf{1}_{\{|a-w|<r\}} = \int_0^r \frac{\mathbf{1}_{\{|a-w|<t\}}}{t}\, \mathrm{d}t.$$

Tonelli interchanges the nonnegative integrals and produces the claimed radial-mass integral. Finiteness follows from the already established first equality; it was not assumed to justify Tonelli. □

One can extract a continuous-source bound directly from the proved identity. For $0 < r < R$, every point in $D(a,r)$ contributes at least $\log(R/r)$ to the radius-R Jensen kernel. Therefore

$$\mu_u(D(a,r)) \le \frac{M_u(a,R) - u(a)}{\log(R/r)}.$$

The numerator is nonnegative, and the denominator is positive. The estimate bounds all source mass in a disk, including singular continuous and area mass, rather than only integer zero multiplicities. It degenerates as $r \uparrow R$ because a source very close to the observation circle can have a large mass and still contribute only a small logarithmic defect. For a single mass m at a distance $d < R$ from the center, the defect is exactly $m \log(R/d)$; choosing d near R shows why no bound independent of the ratio r/R follows from that defect alone.

Even at a pole the circular mean at every positive radius is finite, by local decomposition and the potential mean formula. In logarithmic radius define $F(s) = M_u(a, e^s)$ on the interval of admissible radii. The local decomposition writes it as a constant plus $\int \max\{s, \log|a-w|\}\, \mathrm{d}\nu(w)$. It follows that F is convex and nondecreasing, with the precise one-sided derivatives

$$F'_-(s) = \mu_u(D(a,e^s)), \qquad F'_+(s) = \mu_u(\overline{D(a,e^s)}). \tag{10.1}$$

To check these formulas, take left or right difference quotients of the hinge function $\max(s,c)$. Their limits are respectively $\mathbf{1}_{\{c<s\}}$ and $\mathbf{1}_{\{c\le s\}}$, and every quotient lies between zero and one.

Dominated convergence against the finite localized measure proves the identities. An atom at a contributes slope one per unit mass for all radii. A positive mass on a circle produces the jump between the two one-sided slopes. There are at most countably many such circles in a compact radius interval, because they are disjoint and the measure there is finite.

10.5 Recovering annular mass from curvature in logarithmic radius

The two one-sided derivatives contain more information than convexity alone. For $s < t$ in an admissible logarithmic-radius interval,

$$F'_+(t) - F'_+(s) = \mu_u\{e^s < |w - a| \le e^t\}.$$

This follows by subtracting the masses of the two nested closed disks. The excluded inner circle and included outer circle are part of the formula; replacing both derivatives by an unspecified ordinary derivative would lose those conventions. In distribution notation the same statement is

$$F'' = (w \mapsto \log|w - a|)_*(\mu_u|_{\{w \neq a\}})$$

on the interval in question. For a direct verification, let $\psi \in C_c^\infty(\mathbb{R})$ have support in that interval. Two integrations by parts for the hinge give

$$\int_{\mathbb{R}} \max\{s, c\}\psi''(s)\,\mathrm{d}s = \psi(c).$$

Integrating against the finite localized measure proves the asserted identity. A source at a contributes the linear term $\mu_u(\{a\})s$ and has zero second derivative at every finite s; its location in logarithmic radius is the limiting end $s = -\infty$. There is no missing finite-radius atom in the displayed pushforward.

This description also proves the two-radius formula without requiring a finite value at the center. Each hinge satisfies

$$\max\{t, c\} - \max\{s, c\} = \int_s^t \mathbf{1}_{\{c<\tau\}}\,\mathrm{d}\tau.$$

For a central source the difference is simply $t - s$. All these integrands are nonnegative, so Tonelli gives

$$M_u(a, R) - M_u(a, r) = \int_r^R \frac{\mu_u(D(a, \rho))}{\rho}\,\mathrm{d}\rho, \qquad 0 < r < R.$$

Both circular means are finite. This proof therefore remains valid when $u(a) = -\infty$, unlike a subtraction involving the center value.

For an exact mixture take uniform circle probability $\sigma_{3/5}$ and unit-disk area probability ν, and set

$$\mu = \frac{3}{10}\delta_0 + \frac{2}{5}\sigma_{3/5} + \frac{3}{10}\nu.$$

Its potential and its closed-disk mass are

$$P_\mu(r) = \frac{3}{10}\log r + \frac{2}{5}\log\max\{r, 3/5\} + \frac{3}{10}\begin{cases}(r^2 - 1)/2, & r \le 1,\\ \log r, & r > 1,\end{cases}$$

$$\mu(\overline{D(0, r)}) = \frac{3}{10} + \frac{2}{5}\mathbf{1}_{\{r \ge 3/5\}} + \frac{3}{10}\min\{r^2, 1\} \qquad (r > 0).$$

The jump at $r = 3/5$ is exactly $2/5$, whereas the density change at $r = 1$ creates no jump: area measure gives that circle zero mass. With $s = \log r$,

$$F'' = \frac{2}{5}\delta_{\log(3/5)} + \frac{3}{5}e^{2s}\mathbf{1}_{\{s<0\}}\,ds, \qquad \lim_{s\to-\infty} F'_+(s) = \frac{3}{10}.$$

The finite-radius curvature records the circle and area sources, and the limiting slope records the central point source. Together they recover the entire radial mass distribution in this example.

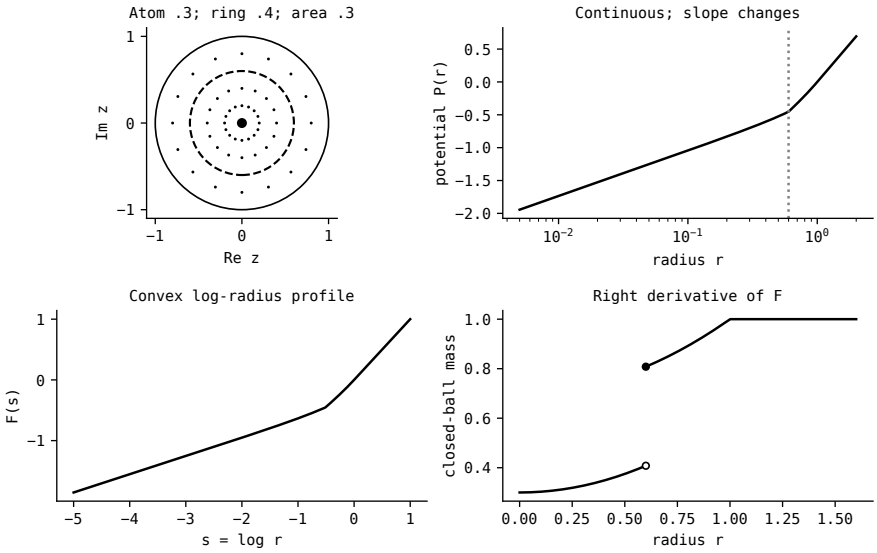


Figure 10.1: One potential with three source types: a central atom, a circle source, and a uniform area source. The logarithmic radius profile has a slope jump of $2/5$ at $\log(3/5)$ and limiting slope $3/10$ as $s \to -\infty$. Closed-disk masses include the circle at its exact radius; the left limit excludes it.

10.6 Holomorphic pullback after regularization

Proposition 10.8 (Holomorphic composition)**.** *If $f : V \to \Omega$ is nonconstant holomorphic on a connected domain and u is subharmonic on Ω, then $u \circ f$ is subharmonic on V.*

Proof. On each relatively compact neighborhood in V, the image under f is compactly contained in Ω. Smooth approximants there satisfy

$$\Delta(u_\varepsilon \circ f)(z) = |f'(z)|^2(\Delta u_\varepsilon)(f(z)) \geq 0.$$

This chain-rule identity also holds at critical points, where the factor $|f'|^2$ is zero. Thus every $u_\varepsilon \circ f$ satisfies the submean inequality. For a fixed sufficiently small circle in V, pass to the limit directly: on a fixed circle the approximants have a common finite upper bound, so reverse Fatou gives

$$(u \circ f)(a) = \lim_{\varepsilon\downarrow 0}(u_\varepsilon \circ f)(a) \leq \limsup_{\varepsilon\downarrow 0} M_{u_\varepsilon\circ f}(a, r) \leq M_{u\circ f}(a, r).$$

For the last step apply Fatou to the common upper bound minus the integrand. The composition is upper semicontinuous. Finally it is not identically $-\infty$: the open mapping theorem makes $f(V)$ open, whereas u is finite almost everywhere and hence cannot be $-\infty$ on that nonempty open image. This proves all parts of the definition. □

In particular subharmonicity is intrinsic on a Riemann surface: two holomorphic coordinate charts have biholomorphic transition maps, and the pullback proposition gives the same condition in either chart. The nonconstant assumption is necessary under our convention; a constant map to a pole would give the excluded function $-\infty$ everywhere.

10.7 An exact smooth approximation to a point mass

The compact-kernel approximation was used for general functions. A different explicit family is useful for seeing how a smooth density concentrates into a point source:

$$q_\varepsilon(z) = \frac{1}{2}\log(|z|^2 + \varepsilon^2), \qquad \varepsilon > 0.$$

It is smooth on the whole plane. For $r = |z| > 0$,

$$q_\varepsilon'(r) = \frac{r}{r^2+\varepsilon^2}, \qquad q_\varepsilon''(r) = \frac{\varepsilon^2 - r^2}{(r^2+\varepsilon^2)^2}, \qquad \Delta q_\varepsilon(r) = \frac{2\varepsilon^2}{(r^2+\varepsilon^2)^2}.$$

The last expression extends continuously to $r = 0$, where it is $2/\varepsilon^2$. Hence

$$d\mu_\varepsilon(z) = \frac{\varepsilon^2}{\pi(|z|^2+\varepsilon^2)^2}\, dA(z), \qquad \mu_\varepsilon(D(0,R)) = \frac{R^2}{R^2+\varepsilon^2}.$$

The mass formula follows by integrating $2\varepsilon^2 r/(r^2+\varepsilon^2)^2$ from zero to R. Letting $R \to \infty$ shows that every measure has total mass one. For fixed $\delta > 0$,

$$\mu_\varepsilon(\mathbb{C} \setminus D(0,\delta)) = \frac{\varepsilon^2}{\delta^2+\varepsilon^2} \longrightarrow 0.$$

Thus μ_ε converges weakly to δ_0: for any bounded continuous g, split the integral of $g(z) - g(0)$ into $|z| < \delta$, controlled by continuity at zero, and its complement, bounded by $2\|g\|_\infty$ times the displayed tail mass. First choose δ small and then ε small. Also $q_\varepsilon \downarrow \log|z|$ as $\varepsilon \downarrow 0$, including value $-\infty$ at zero. This family has an infinite-support density and is not the compactly supported convolution kernel used earlier. Its elementary formula makes both the singular limit and its mass normalization visible.

Investigation 10.9 (The Riesz measure of a quadratic)**.** Find μ_u for $u(z) = a|z|^2 + \mathrm{Re}(bz^3) + c$ with $a \geq 0$, $b \in \mathbb{C}$, and $c \in \mathbb{R}$.

Solution. The cubic real part and the constant are harmonic, while $\Delta(a|z|^2) = 4a$. Therefore $\mu_u = (2a/\pi)dA$. In $D(0,r)$ its mass is $2ar^2$. Jensen–Riesz gives

$$M_u(0,r) - u(0) = \int_0^r \frac{2at^2}{t}\,\mathrm{d}t = ar^2,$$

which also follows by averaging the polynomial directly. This checks both the 2π normalization and the radial integral.

Investigation 10.10 (A harmonic distribution and its point values)**.** Let v equal zero except that $v(0) = 100$. What does its distributional Laplacian determine?

Solution. As an L^1_{loc} class, v is zero and $\Delta v = 0$. Weyl's lemma supplies the harmonic representative $h \equiv 0$. It does not assert that the original arbitrary point value 100 is harmonic. The distributional characterization selects a unique subharmonic representative, also zero, using the canonical area-average condition. Distributions forget sets of area zero; subharmonic point values recover them by the mean inequalities.

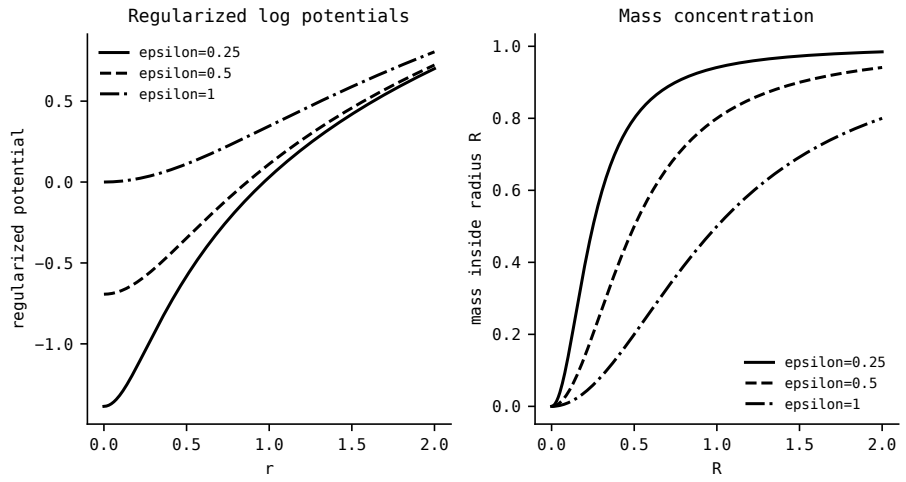


Figure 10.2: Exact smooth point-source regularizations. The density is $\varepsilon^2/[\pi(r^2+\varepsilon^2)^2]$ and the enclosed mass is $R^2/(R^2+\varepsilon^2)$. Different line patterns show different positive ε. The mass remains one while its concentration radius shrinks; the limiting potential is $\log|z|$.

Investigation 10.11 (A circle creates a slope jump)**.** For $u(z) = m\log^+(|z|/R_0)$, $m > 0$, determine the one-sided slopes of $F(s) = M_u(0, e^s)$.

Solution. Here $F(s) = m\max(0, s - \log R_0)$. Both slopes are zero for $s < \log R_0$ and m for $s > \log R_0$. At $s = \log R_0$, the left slope is zero and the right slope is m. Accordingly μ_u assigns mass m to the circle of radius R_0 and no mass to its open interior. This is the distinction between open and closed balls in (10.1).

Investigation 10.12 (An atom forces a pole)**.** If $\mu_u(\{a\}) = m > 0$, prove that $u(a) = -\infty$.

Solution. Local Riesz decomposition contains the summand $m\log|z-a|$. At $z = a$ that summand is $-\infty$; the remaining positive compact-measure potential has a finite upper bound locally, and the harmonic remainder is finite. Thus the whole sum is $-\infty$ at a. The converse need not hold: a negative infinite potential can arise from a sufficiently concentrated non-atomic measure, as the next investigation demonstrates.

Investigation 10.13 (A non-atomic source with an infinite potential)**.** On $0 < |z| < e^{-1}$ put $d\mu(z) = [2\pi|z|^2(\log(1/|z|))^2]^{-1}dA(z)$, and give zero mass to the origin and the complement. Determine its total mass and its potential at zero.

Solution. Polar coordinates and $t = \log(1/r)$ give

$$\mu(\mathbb{C}) = \int_0^{e^{-1}} \frac{dr}{r(\log(1/r))^2} = \int_1^\infty \frac{dt}{t^2} = 1.$$

It is absolutely continuous with respect to area and hence has no atoms. Nevertheless

$$P_\mu(0) = \int_0^{e^{-1}} \frac{\log r}{r(\log(1/r))^2}\,\mathrm{d}r = -\int_1^\infty \frac{dt}{t} = -\infty.$$

Its support is compact after closure, and the positive-potential theorem still gives a locally integrable subharmonic function. A pole of a subharmonic function therefore does not by itself imply an atomic source or an integer multiplicity.

Investigation 10.14 (A concentration radius)**.** For the smooth point-source measure μ_ε, find the radius containing a prescribed mass fraction $0 < \alpha < 1$.

Solution. Solve $R^2/(R^2+\varepsilon^2)=\alpha$:

$$R=\varepsilon\sqrt{\frac{\alpha}{1-\alpha}}.$$

Half the mass lies in radius ε; nine tenths lies in radius 3ε. This exact relation is a concentration scale for the analytic family. It is not a spatial-resolution guarantee for a numerical mesh unless that separate approximation is analyzed.

Investigation 10.15 (Critical points of a pullback)**.** Let $u(z)=|z|^2$ and $f(w)=w^k$, $k\geq 2$. Compute the Laplacian of $u\circ f$ and its value at the critical point.

Solution. The composition is $|w|^{2k}$, and for $w\neq 0$ the radial formula gives $\Delta|w|^{2k}=4k^2|w|^{2k-2}$. Both sides extend continuously to zero, where the value is zero. The chain rule agrees: $|f'(w)|^2(\Delta u)(f(w))=k^2|w|^{2k-2}\cdot 4$. A critical point does not reverse the sign of the Laplacian; it may make the density vanish.

Investigation 10.16 (Harmonic ambiguity and unique sources)**.** Suppose u_1,u_2 are subharmonic and their Riesz measures agree. What can be concluded about their difference?

Solution. The almost-everywhere difference u_1-u_2 is locally integrable and has zero distributional Laplacian. Weyl's lemma gives a harmonic function h equal to that difference almost everywhere. Because u_2+h is subharmonic and equals u_1 almost everywhere, canonical values give $u_1=u_2+h$ at every point. Thus the measure fixes the subharmonic function up to a harmonic term. Boundary data or a normalization are needed to determine that remaining term uniquely.

10.8 Growth at infinity and the total source mass

Local decomposition describes sources on a disk. A global upper bound restricts how much source mass can occur in the entire plane. The useful estimate is obtained by placing a fixed inner disk inside circles of arbitrarily large radius, not by examining a finite sampled annulus.

Proposition 10.17 (Logarithmic growth controls mass)**.** *Let u be subharmonic on $\mathbb{C}$. Suppose that for constants $A\geq 0$ and $B\in\mathbb{R}$,*

$$u(z)\leq A\log^+|z|+B\quad(z\in\mathbb{C}).$$

Then $\mu_u(\mathbb{C})\leq A$. In particular an entire subharmonic function bounded above is constant.

Proof. Choose a with $u(a)$ finite, and fix $r>0$. For $R>r$, the Jensen–Riesz formula and positivity of the measure imply

$$\mu_u(D(a,r))\log(R/r)\leq M_u(a,R)-u(a)\leq A\log^+(R+|a|)+B-u(a).$$

Divide by $\log(R/r)>0$ and let $R\to\infty$. The ratio $\log(R+|a|)/\log(R/r)$ tends to one, while the fixed constants divided by $\log(R/r)$ tend to zero. Therefore $\mu_u(D(a,r))\leq A$ for every r. Continuity from below of the measure gives $\mu_u(\mathbb{C})\leq A$ as these disks exhaust the plane.

If u is bounded above, take $A=0$. Its Riesz measure vanishes, so the local decomposition shows that u is harmonic everywhere. To prove constancy without a lower bound on u, let M be a global upper bound and put $v=M-u+1>0$. The disk Harnack inequality on $D(0,R)$ gives, for fixed z and $R>|z|$,

$$\frac{R-|z|}{R+|z|}\,v(0)\leq v(z)\leq\frac{R+|z|}{R-|z|}\,v(0).$$

Letting $R\to\infty$ yields $v(z)=v(0)$ and hence $u(z)=u(0)$. The argument first removed all source mass and then removed the remaining harmonic freedom. □

For $u = \log|p|$, where p is a polynomial of degree d, the total Riesz mass is exactly d by the zero-counting identity. The elementary polynomial bound $|p(z)| \le C\max(1,|z|)^d$ gives $A = d$ and recovers the matching upper bound. For a general entire subharmonic function, the total measure need not be integral: $\alpha\log^+|z|$ has total mass α for every real $\alpha > 0$.

Proposition 10.18 (The coefficient detected at a point)**.** *For a subharmonic function near a,*

$$\lim_{r\downarrow 0}\frac{M_u(a,r)}{\log r} = \mu_u(\{a\}).$$

This is a statement about circular means, not about every pointwise approach to a.

Proof. Choose a fixed small radius R and write $m = \mu_u(\{a\})$. The one-sided radial derivative formula, or directly the local potential representation, gives

$$M_u(a,R) - M_u(a,r) = \int_r^R \frac{\mu_u(D(a,t))}{t}\,\mathrm{d}t.$$

Changing open to closed balls affects at most countably many radii and not this integral. As $t\downarrow 0$, continuity from above of the finite local measure gives $\mu_u(D(a,t)) \to m$. For each $\eta > 0$ choose $\delta < R$ so that $m \le \mu_u(D(a,t)) \le m+\eta$ for $0 < t < \delta$. Split the radial integral at δ. Its portion from δ to R is a fixed finite constant, while the portion from r to δ lies between $m\log(\delta/r)$ and $(m+\eta)\log(\delta/r)$. Dividing by $\log(1/r)$, letting $r\downarrow 0$, and then $\eta\downarrow 0$ gives the asserted limit. The finite number $M_u(a,R)$ disappears after the same division. □

Rates beyond the atomic coefficient. Write $m(t) = \mu_u(D(a,t))$, $m_0 = \mu_u(\{a\})$, and $q(t) = m(t) - m_0 \ge 0$. The two-radius identity isolates the remainder:

$$M_u(a,r) - m_0\log r = M_u(a,R) - m_0\log R - \int_r^R \frac{q(t)}{t}\,\mathrm{d}t.$$

If $q(t) \le Ct^\alpha$ for $0 < t < R$, where $\alpha > 0$, the integral from zero to R converges. Defining

$$A_R = M_u(a,R) - m_0\log R - \int_0^R \frac{q(t)}{t}\,\mathrm{d}t,$$

one obtains the signed error estimate

$$0 \le M_u(a,r) - m_0\log r - A_R = \int_0^r \frac{q(t)}{t}\,\mathrm{d}t \le \frac{C}{\alpha}r^\alpha.$$

It is a rate for circular means. No assertion about every angular approach to the point follows from this radial mass bound alone.

Slower decay can change the finite remainder while leaving the atomic coefficient unchanged. On $0 < t < e^{-1}$ consider the radial positive measure with a point mass m_0 at zero and area density

$$\frac{c\beta}{2\pi|z|^2[\log(1/|z|)]^{\beta+1}}, \qquad c > 0, \quad \beta > 0.$$

The diffuse density is set equal to zero for $|z| \ge e^{-1}$; the single point $z = 0$ is treated by its specified atom, not by an area density.

Polar integration with $T = \log(1/t)$ gives $m(t) = m_0 + cT^{-\beta}$, and the total diffuse mass is c. Its diffuse potential at $r = e^{-T}$ is therefore exactly

$$-c - c\int_1^T s^{-\beta}\,\mathrm{d}s = \begin{cases} -c - c\log T, & \beta = 1, \\ -c - c\dfrac{T^{1-\beta}-1}{1-\beta}, & \beta \ne 1. \end{cases}$$

The value at $r = e^{-1}$ is $-c$ because every diffuse source lies within that circle. The two-radius formula then proves the displayed expression at every smaller radius. For $\beta > 1$ the diffuse potential tends to the finite value $-c\beta/(\beta - 1)$; for $\beta = 1$ it diverges like $-c \log T$; for $0 < \beta < 1$ it diverges like $-cT^{1-\beta}/(1 - \beta)$. In every case its ratio to $\log r = -T$ tends to zero. Adding $m_0 \log r$ consequently changes the limiting ratio to m_0 and leaves the three remainder regimes distinct. This calculation exhibits exactly what the atomic-coefficient theorem does and does not measure.

A pole can thus have zero logarithmic coefficient. The non-atomic probability density from the preceding investigation has, for $0 < r < e^{-1}$,

$$\mu(D(0, r)) = \frac{1}{\log(1/r)}, \qquad P_\mu(r) = -1 - \log\log(1/r).$$

To verify the second expression, the mass inside radius r contributes $\log r/\log(1/r) = -1$, and the mass between r and e^{-1} contributes $-\int_1^{\log(1/r)} dt/t$. Its potential tends to $-\infty$ at zero, but dividing by $\log r$ gives zero. This example separates a logarithmic atom, a weaker non-atomic pole, and a removable finite value.

Investigation 10.19 (A nonintegral total mass)**.** Can an entire subharmonic function with growth at most $\frac{1}{2}\log^+|z|$ have Riesz mass $1/2$?

Solution. Yes: take $u(z) = \frac{1}{2}\log^+|z|$. Its Riesz measure is one half of uniform circle probability and has mass $1/2$. Such a measure cannot be the zero-counting measure of a nonzero holomorphic function because each zero has positive integer multiplicity. General subharmonic potentials therefore form a strictly larger class than logarithms of holomorphic moduli.

Investigation 10.20 (A pole without a logarithmic atom)**.** For the non-atomic radial density above, compute $\lim_{r\downarrow 0} P_\mu(r)/\log r$ and its radial mass slope.

Solution. Writing $T = \log(1/r) \to \infty$ gives $P_\mu(r)/\log r = (1 + \log T)/T \to 0$. On logarithmic radius $s = \log r$, $P_\mu(e^s) = -1 - \log(-s)$ and its derivative is $-1/s = 1/T$, exactly the enclosed mass. Thus the radial mean diverges to $-\infty$ even though its slope tends to zero; neither condition alone means there is an atom.

Chapter 11

Perron envelopes and harmonic replacement

Source correspondence. Lyubich, Section 10.6, Proposition 10.16, and the boundary family in Section 10.7.

The Poisson formula solves a boundary problem on a disk because the kernel is explicit. Perron's method replaces that formula by an order construction. One collects functions that lie below the prescribed boundary data, raises them inside disks, and takes their supremum. The essential point is that the final supremum becomes harmonic even though a general supremum of continuous functions need not be continuous. This chapter proves that improvement without assuming any regularity of the envelope in advance.

Throughout this chapter, U is a connected domain in a Riemann surface. A coordinate disk $D \Subset U$ means that a chart identifies D with a round Euclidean disk and identifies its closure with a compact subset of the chart. All functions in a Perron family here are finite, continuous and subharmonic. The restriction to continuous competitors is sufficient for continuous Dirichlet data and makes the boundary values used in each disk lift literal. It is not a claim that all useful Perron constructions require continuity.

11.1 The order structure of disk lifting

For $u \in C(U)$ subharmonic and $D \Subset U$, write $L_D u$ for the function which equals u on $U \setminus D$ and equals the Poisson extension of $u|_{\partial D}$ on D. The lifting theorem from the subharmonic development states that

$$u \le L_D u, \qquad L_D u \in C(U), \qquad L_D u \text{ is subharmonic on } U, \qquad \Delta(L_D u) = 0 \text{ on } D.$$

The comparison at the seam is part of this theorem. A piecewise definition with matching boundary values is not, on its own, a subharmonicity proof.

Lemma 11.1 (Monotonicity and idempotence of a lift)**.** *For continuous subharmonic u, v and a coordinate disk $D \Subset U$,*

$$u \le v \Longrightarrow L_D u \le L_D v, \qquad L_D(L_D u) = L_D u.$$

If u is harmonic on D, then $L_D u = u$ on all of U.

Proof. Outside D the first assertion is the assumed inequality. Inside D, both lifted functions are harmonic and their difference has nonpositive boundary values; harmonic comparison gives $L_D u - L_D v \le 0$. For idempotence, the first lift already is the unique harmonic function with

boundary data $u|_{\partial D}$. A second lift has the same boundary data and is equal to the first by uniqueness. The last assertion is precisely the same uniqueness argument applied to the original u.

In disk coordinates $z = c + Rw$, this order statement can also be read directly from the positive kernel:

$$(L_D v - L_D u)(c + Rw) = \frac{1}{2\pi} \int_0^{2\pi} (v - u)(c + Re^{it}) \frac{1 - |w|^2}{|e^{it} - w|^2} \, \mathrm{d}t \geq 0.$$

This is a second proof of the interior inequality; it exposes why a sign-changing integral kernel would not support the same construction. □

Definition 11.2 (Perron family)**.** A nonempty family $\mathcal{P}$ of finite continuous subharmonic functions on U is a Perron family if

$$u, v \in \mathcal{P} \Longrightarrow \max(u, v) \in \mathcal{P}, \qquad u \in \mathcal{P},\ D \Subset U \Longrightarrow L_D u \in \mathcal{P}.$$

Its pointwise envelope is

$$H(z) = \sup\{u(z) : u \in \mathcal{P}\} \in \mathbb{R} \cup \{+\infty\}.$$

Nonemptiness prevents $H = -\infty$: a fixed $u_0 \in \mathcal{P}$ satisfies $H(z) \geq u_0(z) > -\infty$. No upper bound, countability or monotone sequence is imposed in this definition. Those structures must be extracted from the two closure properties.

11.2 A one-point contact proof of the envelope theorem

Theorem 11.3 (Perron's alternative)**.** *For a Perron family on a connected domain U, exactly one of the following holds: H is a finite harmonic function throughout U, or $H \equiv +\infty$.*

Proof. Suppose first that $H(a) < \infty$ at some point a. Choose a coordinate disk $D \Subset U$ containing a. For each $n \geq 1$, the definition of supremum provides $u_n \in \mathcal{P}$ with

$$H(a) - \frac{1}{n} < u_n(a) \leq H(a).$$

Choose any $u_0 \in \mathcal{P}$ and define recursively

$$v_1 = L_D \max(u_0, u_1), \qquad v_{n+1} = L_D \max(v_n, u_{n+1}).$$

Every v_n lies in $\mathcal{P}$, is harmonic on D, and satisfies

$$u_0 \leq v_n \leq v_{n+1} \leq H \quad \text{on } U, \qquad H(a) - \frac{1}{n} < v_n(a) \leq H(a).$$

The inequality $v_n \leq v_{n+1}$ uses both $v_n \leq \max(v_n, u_{n+1})$ and monotonicity of lifting. The one-point upper bound lets the monotone argument be quantified before any continuity of H is known. Put $h_0 = L_D u_0$; monotonicity gives $v_n \geq h_0$ on D. The differences $v_n - h_0$ are nonnegative harmonic functions with

$$0 \leq v_n(a) - h_0(a) \leq H(a) - h_0(a) < \infty.$$

For every compact $K \Subset D$, Harnack comparison therefore supplies a constant $C_{K,a,D}$, independent of n, such that

$$0 \leq v_n(z) - h_0(z) \leq C_{K,a,D}\big(H(a) - h_0(a)\big) \qquad (z \in K).$$

The same estimate applied to $v_m - v_n$ for $m \geq n$ gives

$$\sup_K |v_m - v_n| \leq C_{K,a,D}\big(v_m(a) - v_n(a)\big).$$

Since the increasing real sequence $v_n(a)$ has a finite limit, the right side tends to zero uniformly in $m \geq n$. Thus the harmonic functions are uniformly Cauchy on each compact set. Their limit is harmonic by the local mean-value characterization. These estimates identify both the compact bound and the Cauchy mechanism. The same one-point bound lets the monotone harmonic convergence theorem, Theorem 4.6, apply on D. Thus $v_n \to v$ locally uniformly on D, where v is harmonic and

$$v \leq H, \qquad v(a) = H(a).$$

The missing inequality is $H \leq v$. It cannot be inferred by declaring H upper semicontinuous. Instead fix an arbitrary $u \in \mathcal{P}$ and set

$$w_n = L_D \max(v_n, u).$$

Again $w_n \in \mathcal{P}$, w_n is harmonic on D, and

$$v_n \leq w_n \leq w_{n+1} \leq H, \qquad w_n \geq u \quad \text{on } D.$$

Since $w_n(a) \leq H(a)$, the same convergence theorem gives a finite harmonic limit w. At the distinguished point,

$$H(a) = v(a) \leq w(a) \leq H(a), \qquad w - v \geq 0 \text{ on } D.$$

The nonnegative harmonic function $w - v$ vanishes at an interior point, so the minimum principle makes it identically zero. Consequently

$$u \leq w = v \quad \text{on } D.$$

Since u was arbitrary, $H \leq v$, and hence $H = v$ is finite harmonic on D.

There is also a quantitative way to discharge the universal quantifier over u. Before taking the limit of w_n, apply Harnack comparison directly to the nonnegative harmonic difference $w_n - v_n$. Its value at a satisfies

$$0 \leq w_n(a) - v_n(a) \leq H(a) - v_n(a) < 1/n.$$

Consequently, for the same compact K and every competitor u,

$$u(z) \leq w_n(z) \leq v_n(z) + \frac{C_{K,a,D}}{n} \qquad (z \in K).$$

The constant depends on the geometry and the contact point, and not on u. Taking the supremum over the entire family is therefore legitimate and gives the stronger conclusion

$$0 \leq H(z) - v_n(z) \leq \frac{C_{K,a,D}}{n} \qquad (z \in K). \tag{11.1}$$

This argument proves locally uniform approximation of the envelope before declaring it continuous. When the chart disk is $D(a, R)$ and $K = \{|z - a| \leq r\}$ with $r < R$, one may take

$$C_{K,a,D} = \frac{R + r}{R - r}.$$

For a contact accuracy ε_n in place of $1/n$, the identical proof gives $C_{K,a,D}\varepsilon_n$. The construction proves the existence of such competitors; it supplies no algorithm for finding a competitor from

an arbitrary description of the family. In particular an observed small gap for one chosen function would not replace the quantified bound on every $u \in \mathcal{P}$ used above.

It remains to propagate the conclusion beyond one disk. Let $F = \{z \in U : H(z) < \infty\}$. The local argument shows that any coordinate disk meeting F lies wholly in F. Thus F is open. Its complement is also open: if $z \notin F$ and a small coordinate disk around z met F, that disk would lie in F, a contradiction. Connectedness and $a \in F$ therefore give $F = U$. The local harmonic descriptions agree because they all equal the same pointwise envelope. If no finite point exists, then $H \equiv +\infty$ by definition. These possibilities are disjoint. □

The proof has two distinct uses of monotonicity. First it converts a sequence which approaches the supremum at one point into a harmonic limit. Then it compares every remaining competitor with that limit by a second sequence. A numerical sequence of disk replacements would reproduce neither the universal quantifier over competitors nor the limiting argument by itself.

11.3 The dense-set proof and its exact logical repair

There is another useful construction. Let $X = \{a_j : j \geq 1\}$ be dense in a disk D. If H is finite somewhere in D, select competitors which approach $H(a_j)$ for the first n points to accuracy $1/n$, take their finite maximum, and lift. Include all earlier competitors at every later stage. This gives increasing harmonic $v_n \in \mathcal{P}$ with

$$v_n \leq H, \qquad \lim_n v_n(a_j) = H(a_j) \quad (j \geq 1).$$

Harnack convergence gives a harmonic limit v. For each fixed continuous competitor u, the inequality $u \leq v$ holds on X and therefore on D. Taking the supremum gives $H \leq v$. Conversely, $v_n \leq H$ pointwise gives $v \leq H$ on passing to the limit. This last inequality uses no semicontinuity of H.

Here is a fully indexed construction, including the finiteness needed to make those selections. Fix $a \in D$ with $H(a) < \infty$ and fix $h_0 = L_D u_0$. For an arbitrary $u \in \mathcal{P}$, set $h_u = L_D \max(u, u_0)$. Then $h_u - h_0 \geq 0$ is harmonic and

$$h_u(a) - h_0(a) \leq H(a) - h_0(a), \qquad u \leq h_u.$$

Harnack comparison on compact subsets of D implies

$$H(z) \leq h_0(z) + C_{K,a,D}\big(H(a) - h_0(a)\big) \quad (z \in K).$$

Thus H is finite at every a_j without assuming the conclusion of the envelope theorem. For every pair $n \geq 1$, $1 \leq j \leq n$, select $u_{n,j} \in \mathcal{P}$ with

$$H(a_j) - 2^{-n} < u_{n,j}(a_j) \leq H(a_j).$$

Start with $v_0 = h_0$ and define the triangular recursion

$$v_n = L_D \max\{v_{n-1}, u_{n,1}, \ldots, u_{n,n}\}. \tag{11.2}$$

The finite maximum is allowed by repeated use of the binary closure property. Induction gives $v_n \in \mathcal{P}$, harmonicity on D, and $v_{n-1} \leq v_n \leq H$. For fixed j the explicit tail estimate is

$$0 \leq H(a_j) - v_n(a_j) < 2^{-n} \qquad (n \geq j).$$

The common bound at a makes v_n converge locally uniformly to a harmonic v. Therefore $v(a_j) = H(a_j)$ for every j. Now fix an arbitrary $u \in \mathcal{P}$ and an arbitrary $z \in D$. Choose a sequence $a_{j_k} \to z$ from the dense set. Continuity of the two individual functions u and v gives

$$u(z) = \lim_k u(a_{j_k}) \leq \lim_k H(a_{j_k}) = \lim_k v(a_{j_k}) = v(z).$$

The middle limit exists because $H = v$ on the selected sequence, not because continuity of H has been assumed. Both u and z were arbitrary, hence $H \le v$ everywhere. The opposite inequality follows from $v_n \le H$ at each fixed point. This separates the three logical operations: selecting countably many competitors, obtaining a locally uniform harmonic limit, and then testing every competitor using continuity on a dense set.

Remark 11.4 (An exact model for the contact constant). On the unit disk let $h(z) = \mathrm{Re}(z^2)$ and

$$v_n(z) = h(z) - \frac{1}{n}P(z,1), \qquad P(z,1) = \frac{1-|z|^2}{|1-z|^2}.$$

These functions are harmonic, increase to h, and have contact defect $h(0) - v_n(0) = 1/n$. In the Perron family of all continuous subharmonic functions bounded above by h, each v_n is admitted and the envelope is h. For $0 < r < 1$ the exact extremes are

$$\min_{|z|\le r}(h - v_n) = \frac{1-r}{n(1+r)}, \qquad \max_{|z|\le r}(h - v_n) = \frac{1+r}{n(1-r)}.$$

They occur at $-r$ and r. Thus the compact-disk contact multiplier cannot be reduced for arbitrary nonnegative harmonic differences. The singularity of $P(z,1)$ is at the boundary; every compact disk used here stays a positive distance from it.

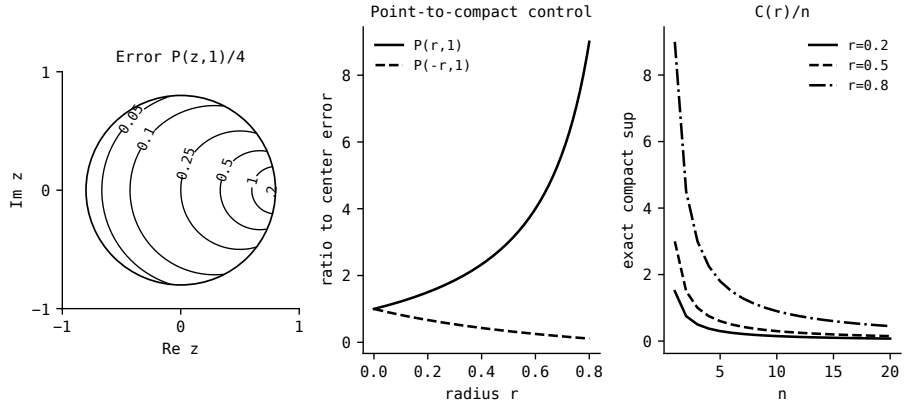


Figure 11.1: Exact contact-defect geometry for $h = \mathrm{Re}(z^2)$ and $v_n = h - P(z,1)/n$. Contours and radial profiles are restricted to compact disks, away from the pole at 1. The extreme gaps $(1 \pm r)/(n(1 \mp r))$ and the $1/n$ convergence profile display the proved Harnack multiplier, rather than a sampled substitute for the universal comparison in (11.1).

In general,

$$\{z : H(z) > t\} = \bigcup_{u \in \mathcal{P}} \{z : u(z) > t\}$$

is open, so a supremum of continuous functions is lower semicontinuous. The conclusion that this particular envelope is harmonic is the output of Perron's theorem, not a preliminary property of arbitrary suprema. For example, $f_n(x) = \min(1, n|x|)$ are continuous on $\mathbb{R}$, but

$$\sup_n f_n(x) = \begin{cases} 0, & x = 0, \\ 1, & x \ne 0, \end{cases}$$

is not upper semicontinuous at 0.

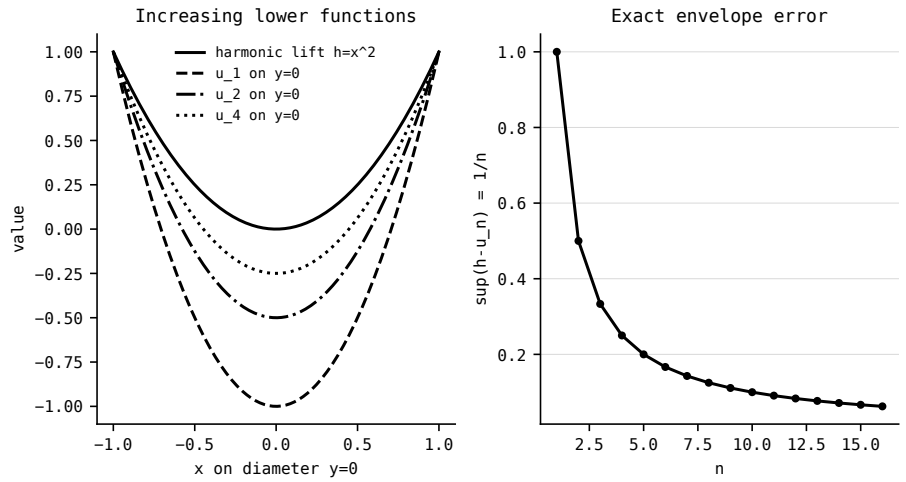


Figure 11.2: An exact increasing family on the unit disk: $u_n = x^2 - y^2 - (1 - x^2 - y^2)/n$, with $\Delta u_n = 4/n$ and common boundary values $\cos(2\theta)$. Every full-disk harmonic lift equals $h = x^2 - y^2$. The vertical gap is $(1 - r^2)/n$, so its uniform bound is exactly $1/n$.

11.4 Five worked investigations

Investigation 11.5 (A finite envelope without a largest competitor)**.** On $\mathbb{D}$, let $\mathcal{P}$ contain all continuous subharmonic u with $\limsup_{z\to\zeta} u(z) \le 0$ for every $\zeta \in \partial\mathbb{D}$. Find its envelope and explain why restricting first to negative constants would not by itself verify the Perron conditions for a larger family.

Solution. The maximum principle gives $u \le 0$ for every competitor, while $u \equiv 0$ is admitted. Thus $H = 0$. If one instead uses only constants $c < 0$, the family is itself Perron, since finite maxima and disk lifting preserve negative constants, and still has envelope 0, which is not a member. This illustrates the distinction between a supremum and a maximum. It does not establish anything about arbitrary added functions: after adding one function, all finite maxima and every disk lift must also be present. The defining closure conditions concern the actual family being used.

Investigation 11.6 (An infinite envelope and connectedness)**.** Construct a Perron family with infinite envelope. Then explain the role of connectedness by considering two disjoint disks.

Solution. The family of all real constant functions on a connected domain is closed under finite maxima and disk lifting; its supremum is $+\infty$ at every point. On $D_1 \sqcup D_2$, take functions which are an arbitrary constant on D_1 and identically zero on D_2. Coordinate disks remain in one component, so this is a Perron family, but its envelope is $+\infty$ on D_1 and 0 on D_2. The alternative in Theorem 11.3 is thus componentwise without connectedness. No chain of coordinate disks can transport a one-point bound across a missing topological connection.

Investigation 11.7 (Quantifying an exact Perron approximation)**.** For the family drawn in Figure 11.2, compute monotonicity, its Laplacian, its boundary trace and its error on $\{|z| \le r\}$.

Solution. Put $h = x^2 - y^2$ and $u_n = h - (1 - x^2 - y^2)/n$. Then

$$\Delta h = 0, \quad \Delta u_n = 4/n > 0, \quad u_{n+1} - u_n = \frac{1 - |z|^2}{n(n+1)} \ge 0.$$

At $|z| = 1$, the correction vanishes and $h(e^{i\theta}) = \cos(2\theta)$. On the closed disk $|z| \le r < 1$ the error satisfies $(1 - r^2)/n \le h - u_n \le 1/n$, with the upper bound attained at 0. The full unit-disk lift

is exactly h by uniqueness. These formulas verify this explicit sequence; the full Perron theorem still requires comparison with every allowed competitor, which this calculation does not replace.

Investigation 11.8 (The contact point cannot be discarded)**.** In the proof of Theorem 11.3, suppose one knew only $w \geq v$ and that $w - v$ is harmonic. Why would this not imply equality?

Solution. On any domain, $v \equiv 0$ and $w \equiv 1$ satisfy both facts. The additional equation $w(a) = v(a)$ makes $w - v$ attain its global minimum zero in the interior. The minimum principle then gives equality everywhere in the disk. Algebraically, the squeeze $v_n(a) \leq w_n(a) \leq H(a)$ and $v_n(a) \to H(a)$ supplies precisely the missing contact. Dropping the upper bound at a would permit arbitrary positive harmonic gaps and destroy the argument.

Investigation 11.9 (A global bound versus a local bound)**.** Suppose a Perron family satisfies $u(a) \leq M$ at one interior point for every $u \in \mathcal{P}$. Prove that its envelope is locally bounded throughout U, even when no global constant upper bound is given.

Solution. Nonemptiness gives a finite lower value, so $u_0(a) \leq H(a) \leq M$. Theorem 11.3 makes H harmonic on U, hence continuous and bounded on every compact $K \Subset U$. Moreover, $u \leq H$ for every competitor. This is a uniform upper bound on K independent of u. There need not be one constant valid on all of U: on the unit disk, the harmonic function $h(z) = \operatorname{Re}\big((1+z)/(1-z)\big) > 0$ is unbounded near 1, while the singleton family $\{h\}$ is Perron and has a finite value at 0. The theorem transports interior control, not boundary boundedness.

Chapter 12

Barriers and attainment of boundary values

Source correspondence. Lyubich, Section 10.7 and Theorem 10.19.

A harmonic envelope is not yet a solution of a boundary problem. Its values must approach the prescribed boundary function at the actual boundary points. A barrier converts a geometric separation from one boundary point into an analytic estimate that forces this convergence. We use positive superharmonic barriers; their negatives are the negative subharmonic barriers in the source. This sign convention makes every comparison coefficient nonnegative.

For the rest of this development, $U \Subset S$ is a connected relatively compact domain in a Riemann surface and $\partial U \neq \varnothing$. Its closure is compact. This hypothesis matters: on a compact surface with $U = S$, the empty boundary imposes no bound on constant competitors. For $g \in C(\partial U)$ put

$$m = \min_{\partial U} g, \qquad M = \max_{\partial U} g,$$

and define

$$\mathcal{P}_U(g) = \left\{ u \in C(U) : u \text{ subharmonic}, \ \limsup_{U \ni z \to a} u(z) \leq g(a) \text{ for every } a \in \partial U \right\}.$$

Boundary limits are taken through U; no trace at an inaccessible point is silently omitted.

12.1 The boundary family is a genuine Perron family

Proposition 12.1 (Admissibility and finiteness)**.** *The family $\mathcal{P}_U(g)$ is Perron, and its envelope H_g is harmonic with $m \leq H_g \leq M$ throughout U.*

Proof. The constant m belongs to the family. Finite maxima preserve both subharmonicity and boundary inequalities, because

$$\limsup_{z \to a} \max(u(z), v(z)) \leq \max\bigl(\limsup_{z \to a} u(z), \limsup_{z \to a} v(z)\bigr) \leq g(a).$$

A disk lift changes a function only on a compact subset of U. Every boundary point has a neighborhood disjoint from that compact subset, so the boundary limsup is unchanged. Thus disk lifting preserves the family.

For completeness, the boundary upper estimate $u \leq M$ can be justified without assuming beforehand that u extends continuously to the boundary. Given $\varepsilon > 0$, each $a \in \partial U$ has a neighborhood in which $u < M + \varepsilon$. A finite collection covers the compact boundary. Its

complement in $\overline{U}$ is a compact subset of U, so if $u > M + \varepsilon$ anywhere, its global maximum above this level is achieved in the interior. The strong subharmonic maximum principle would make u constant on U. Such a constant contradicts the boundary limsup bound. Therefore $u \le M + \varepsilon$, and letting $\varepsilon \downarrow 0$ gives $u \le M$. Perron's theorem now gives a finite harmonic envelope, and the constant competitor gives $H_g \ge m$. □

Definition 12.2 (Strict local barrier)**.** A positive local barrier at $a \in \partial U$ is a finite function b on $\overline{U} \cap N$, where N is an open neighborhood of a, such that b is continuous there, superharmonic on $U \cap N$, and

$$b(a) = 0, \qquad b(z) > 0 \quad (z \in \overline{U} \cap N,\ z \neq a).$$

A point possessing such a barrier is called regular in the barrier sense.

We retain strict positivity on nearby boundary points as well as on interior points. A function vanishing along an entire boundary arc is useful in other comparisons but does not satisfy this definition.

12.2 Globalizing a local barrier

Lemma 12.3 (Local-to-global barrier construction)**.** *A local barrier at a yields a continuous function B on $\overline{U}$, superharmonic on U, such that $B(a) = 0$ and $B > 0$ on $\overline{U} \setminus \{a\}$. It may be chosen constant outside an arbitrarily small coordinate neighborhood on which the local barrier is available.*

Proof. Choose a chart centered at a and $r > 0$ such that the closed coordinate disk $\{|z| \le 2r\}$ lies inside N. On the compact set

$$K = \overline{U} \cap \{r \le |z| \le 3r/2\}$$

the local barrier is strictly positive. If K is nonempty, let $c = \frac{1}{2} \min_K b > 0$; if it is empty, choose any $c > 0$. Define

$$B(z) = \begin{cases} \min\{b(z), c\}, & z \in \overline{U},\ |z| < 3r/2, \\ c, & z \in \overline{U} \text{ outside that disk.} \end{cases}$$

On the annulus $r \le |z| \le 3r/2$ the two descriptions agree because $b \ge 2c$. Hence B is continuous across the artificial seam and is locally constant in a full relative annular neighborhood of it. Inside the disk, a finite minimum of superharmonic functions is superharmonic, since its negative is a finite maximum of subharmonic functions. Outside, $B = c$ is harmonic. These local statements prove superharmonicity on all of U. The only zero is a, because both b and c are positive elsewhere. The compact annular minimum, rather than an unsupported supremum over an open circular arc, is what guarantees a strictly positive gluing level.

The seam can be checked point by point. A point of U with coordinate modulus exactly $3r/2$ has a neighborhood whose part inside that circle lies in $r < |z| \le 3r/2$. There $b \ge 2c > c$, so the interior definition already equals c; the exterior definition equals c as well. A point strictly inside the circle has a neighborhood on which the definition is the single superharmonic function $\min(b, c)$. A point strictly outside has a neighborhood with constant value. These three cases cover U. If K is empty, there is no point of $\overline{U}$ in the intervening closed annulus, so no seam point needs checking. This case does not require assigning a minimum to an empty set.

For later estimates, the construction gives, for each $0 < t < r$,

$$\min_{\overline{U} \setminus \{|z| < t\}} B \ge \min\left(c, \min_{\overline{U} \cap \{t \le |z| \le 3r/2\}} b\right) > 0$$

whenever the inner displayed compact set is nonempty; when it is empty the lower bound is c. Near a the opposite estimate is $0 \le B \le b$. Consequently clipping preserves the local decay toward zero and replaces the distant geometry by a fixed positive level. Division by c produces a barrier equal to one outside the selected neighborhood, with values in $[0,1]$ everywhere. Multiplying any barrier by a positive scalar changes both its distant lower bound and its local upper bound by that scalar; their ratio in the boundary comparison is unchanged. □

Lemma 12.4 (Restriction to a smaller domain)**.** *If $V \subset U$ is a domain and $a \in \partial V \cap \partial U$ is regular for U, then a is regular for V.*

Proof. Restrict the same local barrier to $\overline{V} \cap N$. Since $\overline{V} \subset \overline{U}$, continuity and strict positivity persist; superharmonicity is local and persists on the open subset $V \cap N$. The value at a is still zero. No smoothness of either boundary is used. □

12.3 A two-sided estimate with all constants visible

Theorem 12.5 (Boundary attainment at a regular point)**.** *If $a \in \partial U$ has a local barrier and $g \in C(\partial U)$, then*

$$\lim_{U \ni z \to a} H_g(z) = g(a).$$

Proof. Use Lemma 12.3 to obtain $B \in C(\overline{U})$. Fix $\varepsilon > 0$. Continuity of g gives a relative neighborhood N_a of a in ∂U on which

$$|g(\xi) - g(a)| < \varepsilon.$$

If $\partial U \setminus N_a$ is nonempty, compactness and strict positivity give

$$c_\varepsilon = \min_{\partial U \setminus N_a} B > 0.$$

Choose

$$A_\varepsilon = \frac{2\|g\|_\infty + \varepsilon}{c_\varepsilon} \ge 0.$$

If the complementary set is empty, take $A_\varepsilon = 0$. Consider

$$\ell(z) = g(a) - \varepsilon - A_\varepsilon B(z), \qquad q(z) = g(a) + \varepsilon + A_\varepsilon B(z).$$

The first is continuous subharmonic, the second continuous superharmonic. On N_a, positivity of B gives $\ell \le g$ and $q \ge g$. On the remaining boundary,

$$\ell \le g(a) - \varepsilon - (2\|g\|_\infty + \varepsilon) \le -\|g\|_\infty \le g,$$

while $q \ge g(a) + \varepsilon + 2\|g\|_\infty + \varepsilon \ge \|g\|_\infty \ge g$. Thus $\ell \in \mathcal{P}_U(g)$ and $\ell \le H_g$. For every competitor u, the continuous subharmonic function $u - q$ has nonpositive boundary limsup. Boundary comparison gives $u \le q$; taking suprema yields $H_g \le q$. We have proved the quantitative estimate

$$|H_g(z) - g(a)| \le \varepsilon + A_\varepsilon B(z) \qquad (z \in U).$$

To spell out the limit quantifiers, let $\eta > 0$ be the desired final error and run the construction with $\varepsilon = \eta/2$. If its coefficient is zero, the error is at most $\eta/2$ throughout U. Otherwise continuity and $B(a) = 0$ give a neighborhood W of a on which $B(z) < \eta/(2A_{\eta/2})$. Every $z \in W \cap U$ then satisfies

$$|H_g(z) - g(a)| \le \eta/2 + A_{\eta/2} B(z) < \eta.$$

The neighborhood is selected after the boundary tolerance and barrier coefficient; neither the approach direction nor a chosen sequence of interior points enters its definition. This is the full boundary limit, including approaches through different local components of U. Since $B(z) \to 0$ as $z \to a$, the limsup of the left side is at most ε. Letting $\varepsilon \downarrow 0$ proves the assertion. □

The estimate separates two kinds of control. The tolerance ε comes from continuity of the boundary data; $B(z)$ and c_ε come from geometry. One should not read it as a uniform power-law convergence rate unless the barrier and the boundary modulus supply such a rate.

12.3.1 A boundary modulus with the geometric denominator exposed

Fix a compatible metric d on the compact closure and a global barrier B at a. Define the boundary oscillation, distant barrier level and local barrier maximum by

$$\begin{aligned}\omega_a(s) &= \sup\{|g(\xi) - g(a)| : \xi \in \partial U,\ d(\xi, a) \le s\},\\ \sigma_a(s) &= \min\{B(\xi) : \xi \in \partial U,\ d(\xi, a) \ge s\},\\ \beta_a(t) &= \sup\{B(z) : z \in \overline{U},\ d(z, a) \le t\}.\end{aligned}$$

Use $\sigma_a(s)$ only when its defining set is nonempty, and put $M_a = \sup_{\partial U} |g - g(a)|$. Compactness and strictness give $\sigma_a(s) > 0$ for $s > 0$, while continuity gives $\omega_a(s) \to 0$ and $\beta_a(t) \to 0$ as $s, t \downarrow 0$. For a fixed admissible s, the boundary inequalities

$$|g(\xi) - g(a)| \le \omega_a(s) + \frac{M_a}{\sigma_a(s)} B(\xi) \qquad (\xi \in \partial U)$$

hold by dividing the boundary into $d(\xi, a) \le s$ and its complement. The same lower and upper comparison functions used in the theorem give

$$\sup_{U \cap \{d(z,a) \le t\}} |H_g(z) - g(a)| \le \omega_a(s) + M_a \frac{\beta_a(t)}{\sigma_a(s)}. \tag{12.1}$$

One may take the infimum over all admissible s on the right. If the whole boundary lies within distance s, the simpler constant comparison gives the bound $\omega_a(s)$ directly. This formula records precisely which boundary modulus and which geometric lower bound are needed to extract a rate; continuity alone does not provide a power exponent.

For an explicit normalization, let $U = \mathbb{D}$, $a = 1$ and $B(z) = 1 - \operatorname{Re} z$. On the boundary $B(\xi) = |\xi - 1|^2/2$, and throughout the closed disk $0 \le B(z) \le |z - 1|$. For $0 < s \le 2$ and $0 < t \le 2$ this gives

$$\sigma_1(s) = s^2/2, \qquad \beta_1(t) \le t, \qquad |H_g(z) - g(1)| \le \omega_1(s) + \frac{2M_1|z - 1|}{s^2}.$$

If $\omega_1(s) \le Ls^\gamma$ with $0 < \gamma \le 1$, choose $s = t^{1/(\gamma+2)}$ for small t to obtain

$$\sup_{|z-1| \le t} |H_g(z) - g(1)| \le (L + 2M_1) t^{\gamma/(\gamma+2)}.$$

This elementary barrier estimate is deliberately nonoptimal; the sharper Poisson estimates earlier in the volume use more information about the kernel. The computation is useful because it verifies every scale and denominator in a general method that remains available when no explicit kernel is known.

12.4 Upper families and exact comparison enclosures

The lower-family construction has a dual description. Define

$$\mathcal{Q}_U(g) = \left\{ q \in C(U) : q \text{ superharmonic},\ \liminf_{z \to a} q(z) \ge g(a) \text{ for every } a \in \partial U \right\}.$$

If $u \in \mathcal{P}_U(g)$ and $q \in \mathcal{Q}_U(g)$, then $u - q$ is subharmonic and has nonpositive boundary limsup, so $u \le q$. Consequently the lower and upper envelopes satisfy

$$\sup_{u \in \mathcal{P}_U(g)} u \le \inf_{q \in \mathcal{Q}_U(g)} q.$$

When the boundary is regular, the common Dirichlet solution itself belongs to both families, and therefore

$$E_U g = \sup \mathcal{P}_U(g) = \inf \mathcal{Q}_U(g).$$

The equality is a theorem about the continuous problem; it does not follow merely from taking many competitors numerically.

Proposition 12.6 (Certified analytic sandwiches)**.** *On a regular domain, let $u \in \mathcal{P}_U(g)$ and $q \in \mathcal{Q}_U(g)$. Then*

$$u \le E_U g \le q.$$

If $q - u \le 2\varepsilon$ on a set $K \subset U$, their midpoint satisfies

$$\sup_K \left| E_U g - \frac{u+q}{2} \right| \le \varepsilon.$$

Proof. The lower bound follows because u is one of the functions whose supremum is $E_U g$. The upper bound follows by boundary comparison, or by the upper envelope identity. At each $z \in K$, the interval $[u(z), q(z)]$ has midpoint $(u(z) + q(z))/2$ and half-length $(q(z) - u(z))/2 \le \varepsilon$. Membership in this interval gives the displayed estimate pointwise; taking the supremum completes the proof. The size of K, its shape and whether it is finite play no role once the inequalities are proved everywhere on it. □

An especially useful case starts with an explicitly harmonic approximation p continuous on the closure. If a rigorous boundary estimate gives $|p - g| \le \varepsilon$ on all of ∂U, then

$$p - \varepsilon \in \mathcal{P}_U(g), \qquad p + \varepsilon \in \mathcal{Q}_U(g), \qquad |p - E_U g| \le \varepsilon \text{ on } \overline{U}.$$

A sampled estimate at finitely many boundary points is not the stated hypothesis. To replace a global boundary bound by samples, one needs an additional modulus of continuity or an analytic interpolation remainder. For example, if the boundary error is known to be L-Lipschitz in an arc parameter and every parameter lies within distance δ of a sampled point, then sampled error at most η implies the global bound $\eta + L\delta$. The resulting harmonic error is at most that same quantity. The Lipschitz bound is a separate proof obligation, not a consequence of a visually smooth graph.

The same sandwich construction explains the role of a barrier in boundary attainment: the functions $g(a) - \varepsilon - A_\varepsilon B$ and $g(a) + \varepsilon + A_\varepsilon B$ are explicit lower and upper competitors whose gap tends to 2ε at a. They do not compute the solution throughout the domain, but they force its unique limiting boundary value.

12.5 Five worked investigations

Investigation 12.7 (The exact disk barrier)**.** For $a \in \partial\mathbb{D}$, prove that $B_a(z) = 1 - \operatorname{Re}(\overline{a}z)$ is a global barrier and compare it with $|z - a|^2$.

Solution. The function is harmonic and continuous on the closed disk. Since $\operatorname{Re}(\overline{a}z) \le |z| \le 1$, it is nonnegative, with equality only when $z = a$. Moreover,

$$|z - a|^2 = 1 + |z|^2 - 2\operatorname{Re}(\overline{a}z) = 2B_a(z) - (1 - |z|^2) \le 2B_a(z).$$

Thus $B_a \geq \frac{1}{2}|z-a|^2$. The squared distance itself is not a positive superharmonic barrier: $\Delta|z-a|^2 = 4 > 0$. Sign and Laplacian are both part of the definition; positivity and a single zero are insufficient.

Investigation 12.8 (Why a boundary-zero Green function is not a point barrier)**.** Suppose G is positive in U and tends to zero at every point of ∂U. Explain why G is not automatically the strict barrier of this chapter.

Solution. If $a \neq b$ are nearby boundary points, a continuous boundary extension has $G(a) = G(b) = 0$. A strict barrier at a must be positive at b, so that condition fails. In addition, a Green function has an interior pole and is not finite throughout U. A local argument away from the pole may use G as a building block, but must create a function which separates a from the rest of the local boundary. The conclusion that a domain is regular cannot be justified merely by pointing to the zero boundary values and ignoring the strictness clause.

Investigation 12.9 (An explicit modulus from a barrier)**.** Suppose $|g(\xi) - g(a)| \leq L|\xi - a|^\beta$ and a global barrier satisfies $B \geq c_0 r^\gamma$ on $\partial U \setminus B(a, r)$, while $B(z) \leq C_0|z-a|^\alpha$. Derive a power bound by choosing r.

Solution. For small r, repeat the comparison with $\varepsilon = Lr^\beta$ and a coefficient bounded by $C_1 r^{-\gamma}$, where C_1 depends on the fixed data norm and c_0. It gives

$$|H_g(z) - g(a)| \leq Lr^\beta + C_1 C_0 r^{-\gamma}|z-a|^\alpha.$$

Balancing the powers by $r = |z-a|^{\alpha/(\beta+\gamma)}$ yields an upper bound $C|z-a|^{\alpha\beta/(\beta+\gamma)}$ for sufficiently small $|z-a|$. This rate is an estimate supplied by these particular hypotheses; it need not be optimal. Without a lower bound away from a, the denominator in the barrier coefficient could deteriorate faster than any power.

Investigation 12.10 (Why compact closure appears twice)**.** Identify two places where relative compactness is used in the boundary construction, and state what breaks on an unbounded planar domain.

Solution. First, finitely many boundary neighborhoods and a compact interior remainder turn boundary limsup control into a maximum-principle bound. Second, the complement of a boundary neighborhood has a positive minimum of the global barrier. On an unbounded domain, a harmonic function can escape the boundary control at infinity. On the upper half-plane, $u(z) = \operatorname{Im} z$ is positive harmonic and has zero boundary trace on the real line, so zero finite-boundary data do not imply uniqueness among all continuous harmonic functions. A condition at infinity or a boundedness class must be imposed. Relative compactness packages that missing end condition into the setup.

Investigation 12.11 (Boundary data ordered only near one point)**.** If $g_1 \leq g_2$ near a regular point a, must $H_{g_1} \leq H_{g_2}$ throughout U? What conclusion always follows at a?

Solution. Global order need not hold: on the disk choose $g_1 = 0$ and a continuous g_2 which is positive near a but has negative average. Then $H_{g_2}(0) < 0 = H_{g_1}(0)$ by the mean value formula. Nevertheless, regularity gives

$$\lim_{z \to a}(H_{g_2}(z) - H_{g_1}(z)) = g_2(a) - g_1(a) \geq 0.$$

Boundary attainment is local in the boundary variable, whereas order in the whole domain requires comparison of the data on the whole boundary. The same distinction underlies localized perturbation estimates using harmonic measure later in the volume.

Chapter 13

Geometric criteria for regular boundary points

Source correspondence. Lyubich, Example 10.17 and Exercise 10.18.

A regularity criterion is useful only when its geometric hypotheses can be checked. Smooth one-sided boundaries give exterior cones and explicit power barriers. A Jordan arc can have no tangent and no exterior cone; its regularity uses a different route. We distinguish these routes, identify the classical topological and conformal inputs in the second, and retain the possibility that a domain approaches a slit from both sides.

13.1 A strict power barrier in a wider wedge

Proposition 13.1 (Exterior-sector criterion)**.** *Choose a complex coordinate with $a = 0$. Suppose, after a rotation, that for some $R > 0$ and $0 < \theta_0 < \pi$,*

$$\overline{U} \cap \{0 < |z| < R\} \subset \{z : |\arg z| \le \theta_0\}.$$

Then a is regular. For any $\theta_0 < \beta < \pi$, a local barrier is

$$b(re^{i\theta}) = r^{\alpha}\cos(\alpha\theta), \qquad \alpha = \frac{\pi}{2\beta}.$$

Proof. The sector $|\arg z| < \beta$ has a single-valued holomorphic logarithm. Define $z^{\alpha} = \exp(\alpha \log z)$ there; its real part b is harmonic. For every point in the indicated closed portion of U,

$$0 < \cos(\alpha\theta_0) \le \cos(\alpha\theta) \le 1,$$

because $|\alpha\theta| \le \alpha\theta_0 < \pi/2$. Therefore

$$c\,|z|^{\alpha} \le b(z) \le |z|^{\alpha}, \qquad c = \cos\left(\frac{\pi\theta_0}{2\beta}\right) > 0.$$

Set $b(0) = 0$. These bounds give continuity at the vertex and strict positivity at every other point of the relative closure. Elsewhere continuity follows from the holomorphic formula. The function is harmonic, hence superharmonic, on U near the vertex. It satisfies the barrier conditions exactly. □

The deliberate inequality $\theta_0 < \beta$ is essential for this strict barrier. If one used $\beta = \theta_0$, the cosine would vanish along both boundary rays, leaving more than one zero in the relative closure. The larger analytic wedge keeps the physical domain away from those zero rays. This widening is possible precisely because a nontrivial exterior sector leaves an angular gap.

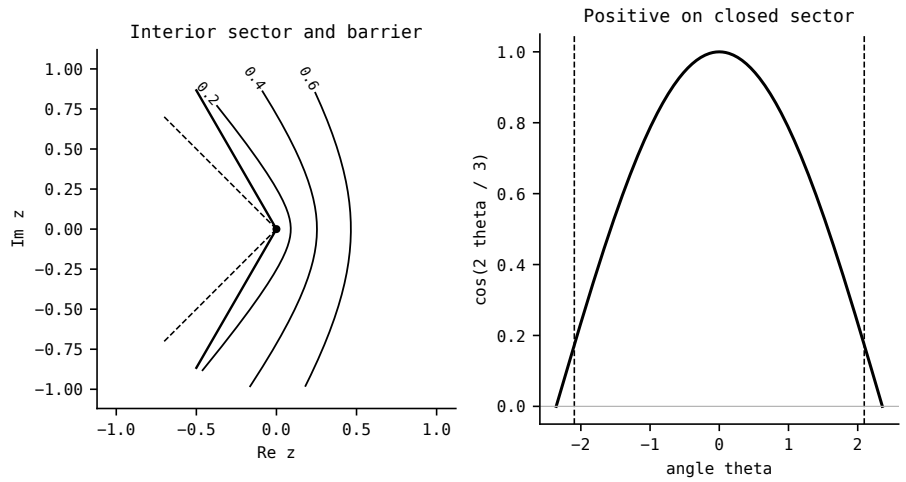


Figure 13.1: A physical sector with $|\theta| \le 2\pi/3$ lies strictly inside the analytic wedge $|\theta| < 3\pi/4$. The harmonic barrier $b = r^{2/3}\cos(2\theta/3)$ has positive lower angular factor $\cos(4\pi/9)$ on the physical sector. Its contour geometry displays both radial decay and separation from the zero rays.

Corollary 13.2 (One-sided smooth boundary)**.** *Suppose in a neighborhood of a the boundary is a C^1 embedded arc with nonzero tangent, and U lies locally on one side of that arc. Then a is regular. In particular every boundary point of a bounded planar domain with C^1 boundary is regular.*

Proof. Translate and rotate so that $a = 0$ and the tangent is horizontal. The implicit function theorem writes the arc as $y = f(x)$, where $f(0) = f'(0) = 0$. Shrinking the neighborhood, for a prescribed $\eta > 0$ we have $|f(x)| \le \eta|x|$. If U is the upper side, the open cone

$$\{(x, y) : y < -2\eta|x|\}$$

near the origin is disjoint from $\overline{U}$ except at the vertex. The complement of a slightly smaller such cone is a closed sector of angle strictly less than 2π containing $\overline{U}$ locally. Rotate its bisector and apply Proposition 13.1. The lower-side case is obtained by reflection. Boundary regularity for a C^1 domain is local, so this argument applies at each point independently. □

13.2 Jordan arcs without an assumed tangent

We now state the inputs used for nonsmooth planar arcs. The local Jordan–Schoenflies theorem straightens an embedded planar arc by a homeomorphism in a neighborhood of each interior point of the arc. In particular one can choose a small topological rectangle cut by the arc into two Jordan domains. The Riemann mapping theorem maps each Jordan domain conformally to $\mathbb{D}$, and the Jordan-domain form of Carathéodory's theorem extends that map to a homeomorphism of closures. These are classical inputs, consistent with the preceding volumes; this chapter does not derive them from the Perron theorem that it is currently using.

A topological straightening alone does not preserve harmonicity. We use it to construct Jordan domains, then use conformal maps to transport Poisson solutions. This distinction prevents the invalid step of composing a harmonic function with an arbitrary homeomorphism.

Theorem 13.3 (Jordan-arc regularity, including two-sided slits)**.** *Let $a \in \partial U$. Suppose in a coordinate neighborhood of a the boundary is exactly an embedded open Jordan arc containing a as an interior point. Then a is regular, whether U occupies one or both local sides of the arc.*

Proof. Use local Jordan–Schoenflies straightening to choose a topological rectangle Q compactly contained in the coordinate neighborhood. The distinguished point is away from its outer boundary, and the given arc divides Q into two Jordan domains D_+ and D_-. More precisely, in straightened coordinates this is the square $(-1,1)^2$ cut by the horizontal diameter; the inverse homeomorphism gives the desired geometry. The common boundary portion is the arc Γ, and $a \in \Gamma$.

Each $D_\pm$ is either entirely inside U or entirely outside it. Indeed, there is no point of ∂U in $D_\pm$, so $U \cap D_\pm$ and its complement are relatively open; connectedness of $D_\pm$ leaves only those two possibilities. At least one side is occupied because $a \in \partial U$.

In the original complex coordinate define the continuous boundary function $\rho(\xi) = |\xi - a|^2$ on the closure of Q. For each occupied side take a conformal map $F_\pm : D_\pm \to \mathbb{D}$ extending homeomorphically to the boundary. Let

$$g_\pm(e^{it}) = \rho\big(F_\pm^{-1}(e^{it})\big), \qquad b_\pm(z) = \frac{1}{2\pi}\int_0^{2\pi} g_\pm(e^{it}) P(F_\pm(z), e^{it})\,\mathrm{d}t.$$

The disk Dirichlet theorem and conformal invariance make $b_\pm$ harmonic on $D_\pm$ and continuous on its closure, with boundary values exactly ρ. The boundary data are nonnegative and not identically zero; positivity of the Poisson kernel gives $b_\pm > 0$ in the domain. On its boundary the only zero is a.

If only one side is occupied, use its $b_\pm$ as the local barrier. If both are occupied, define $b = b_+$ on D_+ and $b = b_-$ on D_-, and set $b = \rho$ on Γ. The two continuous extensions agree on the whole common arc because they were assigned the same geometric boundary data. For an explicit continuity check, fix $q \in \Gamma$ and $\varepsilon > 0$. Continuity of b_+ on its closure gives a neighborhood W_+ of q such that $|b_+(z) - \rho(q)| < \varepsilon$ for points of that closure in W_+. The analogous neighborhood W_- exists for the other occupied side. Intersect them with a neighborhood on which $|\rho(\xi) - \rho(q)| < \varepsilon$ along Γ. Every point of the relative closure in the resulting neighborhood belongs to an occupied side closure or to Γ, so its assigned value lies within ε of $\rho(q)$. If just one side is occupied, the same argument uses that side only. The check applies at every q on the interior arc, not just at its distinguished zero. At a it gives the required zero limit even along a sequence alternating between the two sides.

The harmonicity assertion has a different quantifier: for every $z \in U \cap Q$, choose a small coordinate disk about z which misses Γ and the outer boundary of Q. It lies entirely in one occupied side, where $b = b_\pm$ is harmonic. There is no need to compare the normal derivatives from opposite sides at points of Γ, since none of those points belongs to the open set on which harmonicity is asserted. Thus b is continuous on the relative closure of $U \cap Q$, is harmonic on each local component of $U \cap Q$, and is strictly positive except at a. Harmonicity is a local property, so it holds on all of $U \cap Q$; there is no equation to impose across the removed arc, which is outside U. This is a strict local barrier. □

The matching boundary data in the two-sided case are a substantive step. Two independently normalized functions $1 - \mathrm{Re}(\overline{F_\pm(a)}F_\pm(z))$ need not agree at other points of the common arc. They would give one-sided limits which might not assemble into a continuous function on the relative closure. Prescribing ρ first resolves that issue without any differentiability of the arc.

13.2.1 An exact slit model and its curved conformal image

The distinction between a removed seam and an interior seam can be made visible in formulas. In a coordinate disk $|w| < 1$, with $w = x + iy$, remove the real diameter and define on the two remaining half-disks

$$b(w) = x^2 - y^2 + 2|y|, \qquad b_+(w) = x^2 - y^2 + 2y, \qquad b_-(w) = x^2 - y^2 - 2y.$$

Each side is harmonic because

$$\Delta(x^2 - y^2) = 2 - 2 = 0, \qquad \Delta(\pm 2y) = 0.$$

For $|y| < 1$,

$$b(w) = x^2 + |y|(2 - |y|) \geq x^2 + |y| \geq 0, \qquad b(w) = 0 \iff w = 0.$$

Both boundary traces on the removed diameter are exactly $b(x) = x^2$. Thus this is an explicit strict local barrier for a domain occupying both sides. It has been constructed with common trace first; no independently chosen conformal normalization needs to match afterwards.

If one instead fills the diameter and asks whether the same continuous formula is superharmonic on the whole disk, the answer changes. Its one-sided derivatives are

$$\partial_y b(x, 0+) = 2, \qquad \partial_y b(x, 0-) = -2.$$

Distributionally $\partial_y^2 |y| = 2\delta_{\{y=0\}}$, so

$$\Delta b = 4\delta_{\{y=0\}} \tag{13.1}$$

in the filled disk. Here the line measure acts on a compactly supported test function φ by $\int \varphi(x, 0)\, \mathrm{d}x$. The sign is positive, so the filled-in function is subharmonic and is not superharmonic. On the slit domain the line is absent, and the same function is harmonic. Matching values therefore proves continuity across a proposed interface, while an interior differential inequality also depends on the derivative jump.

To obtain an exact curved picture rather than an arbitrary distorted cartoon, use the holomorphic map

$$F(w) = w + \frac{i}{5} w^2.$$

For $w_1, w_2 \in \overline{\mathbb{D}}$,

$$F(w_1) - F(w_2) = (w_1 - w_2)\left(1 + \frac{i}{5}(w_1 + w_2)\right), \qquad \left|1 + \frac{i}{5}(w_1 + w_2)\right| \geq \frac{3}{5}.$$

Thus F is injective on the closed disk, with nonzero derivative in a neighborhood of it. It maps the real diameter to the parabolic arc $F(t) = t + it^2/5$. On the two image sides the function $\widetilde{b} = b \circ F^{-1}$ is harmonic by conformal invariance, and both side traces at $F(t)$ equal t^2. It is positive except at $F(0)$. This explicit model illustrates the gluing mechanism for one smooth curved arc. The preceding Jordan theorem applies much more generally, using closure continuity rather than a formula for F or a derivative bound.

13.3 Removing an interior disk

Proposition 13.4 (Regularity survives an interior circular hole)**.** *Suppose $U \Subset S$ has nonempty regular boundary. Let D be a coordinate disk with $\overline{D} \Subset U$ and whose closed disk lies in a larger coordinate disk compactly contained in U. Then $V = U \setminus \overline{D}$ is connected and has regular boundary*

$$\partial V = \partial U \sqcup \partial D.$$

Proof. The boundary identity follows from $\overline{D} \Subset U$. At a point of ∂U, restrict an existing local barrier using Lemma 12.4. At a point of ∂D, the new domain lies on the exterior side of a smooth circle, so Corollary 13.2 applies.

To check connectedness, join any two points of V by a path in U; domains in a surface are path connected because they are connected and locally path connected. Choose concentric coordinate

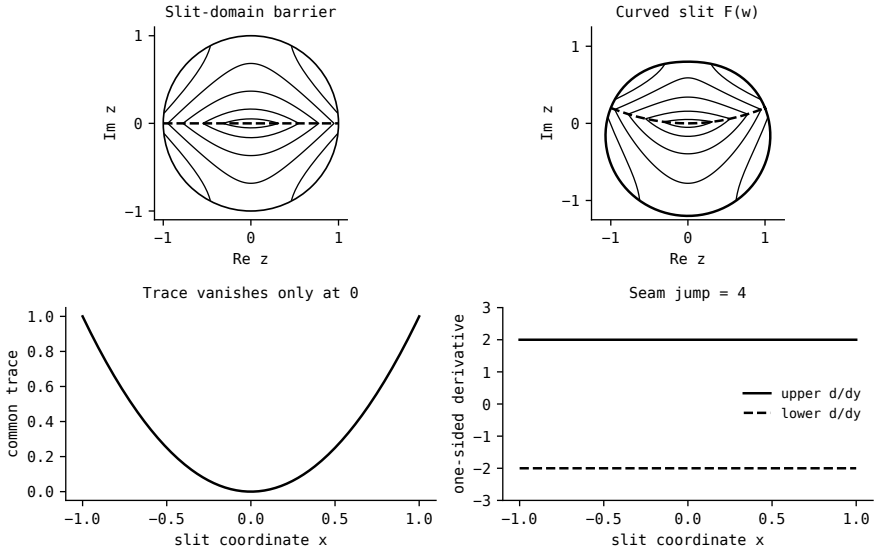


Figure 13.2: A two-sided barrier with matching slit traces: $b = x^2 - y^2 + 2|y|$ and its conformal transport under $F(w) = w + iw^2/5$. The geometric and contour panels show the two occupied sides; the trace and derivative panels record $b(x, 0) = x^2$ and the derivative jump 4. Harmonicity holds off the removed seam. Filling that seam gives the positive line measure in (13.1), not a superharmonic extension across an interior interface.

disks $D \Subset D' \Subset U$. If the path enters $\overline{D}$, replace the portion from its first entry into $\overline{D'}$ to its last exit by a path in $\overline{D'} \setminus \overline{D}$, connecting the same entry and exit points. This annular set is path connected. If an endpoint already lies in $D' \setminus \overline{D}$, use that endpoint as the corresponding endpoint of the replacement. Formally, take the first and last times at which the path meets a concentric closed circle whose radius is strictly between the radii of D and D'; continuity supplies those times whenever both endpoints are outside that circle. The endpoint cases use radial segments in the annulus. Before the first and after the last selected time, the path avoids D. The replacement also avoids $\overline{D}$. It is a path in V with the original endpoints, proving connectedness. □

13.4 Five worked investigations

Investigation 13.5 (A reentrant corner)**.** For the sector $0 < r < 1$, $|\theta| < 3\pi/4$, construct a strict local barrier at the origin with an explicit exponent.

Solution. Choose the larger analytic half-angle $\beta = 7\pi/8$. Then $\alpha = \pi/(2\beta) = 4/7$ and

$$b(re^{i\theta}) = r^{4/7}\cos(4\theta/7), \qquad b \geq r^{4/7}\cos(3\pi/7) > 0.$$

The cosine lower bound remains positive on the closure of the physical sector. The reentrant angle $3\pi/2$ is larger than π, so an interior half-plane model would be insufficient, but the wider-wedge construction still applies. The artificial circular boundary $r = 1$ is irrelevant to this local barrier; choose a neighborhood of radius less than one.

Investigation 13.6 (Why a Jordan arc need not give an exterior cone)**.** Explain why the proof of Theorem 13.3 does not require a power-law rate for $b(z) \to 0$ and why that matters for a highly distorted arc.

Solution. The closure homeomorphism $F_\pm$ is continuous at a, so $F_\pm(z) \to F_\pm(a)$ and the Poisson solution tends to its prescribed boundary value zero. No bound such as $|F_\pm(z) - F_\pm(a)| \leq C|z-a|^\alpha$

was used or proved. A Jordan arc may turn or oscillate so sharply near a point that no fixed exterior cone is available. The topological construction still provides a Jordan domain and continuity of its conformal boundary map. Hence it gives qualitative regularity while leaving quantitative moduli unspecified. A displayed smooth cartoon of the arc would not justify adding a cone or an unproved Hölder exponent.

Investigation 13.7 (A puncture is an irregular boundary point)**.** Let $U = \mathbb{D} \setminus \{0\}$ and prescribe $g = 0$ on $\partial\mathbb{D}$, $g(0) = 1$. Show that these are continuous data on ∂U but admit no continuous harmonic solution on $\overline{U}$.

Solution. The boundary is the disjoint union of the unit circle and the isolated point 0, so the assigned function is continuous on that compact set. Any continuous-on-closure solution would be bounded near 0. The removable-singularity theorem for bounded harmonic functions, Theorem 5.1, would extend it harmonically to the whole disk. Its zero unit-circle boundary data would force it to be identically zero, contradicting the value one at the puncture. Thus 0 cannot have a strict barrier: otherwise boundary attainment and regularity of the outer circle would provide the forbidden solution. A boundary point being isolated is therefore not a favorable regularity condition in planar potential theory.

Investigation 13.8 (Monotonicity of regularity is one-sided)**.** Use the restriction lemma to show that a domain inside a disk is regular at any common boundary point on the circle. Explain why enlarging the domain does not give the same conclusion.

Solution. At $a \in \partial\mathbb{D}$, the barrier $1 - \mathrm{Re}(\overline{a}z)$ restricts to every $V \subset \mathbb{D}$ with $a \in \partial V$, giving regularity there. Enlargement need not preserve the geometry that made this function positive: outside the supporting tangent line its real-part expression can be negative. More concretely, a small disk internally tangent at 0 to the punctured unit disk has 0 as a regular boundary point, while the larger domain $\mathbb{D} \setminus \{0\}$ has 0 irregular by the preceding investigation. One can take the small disk centered at $1/4$ with radius $1/4$.

Investigation 13.9 (Two sides of a slit)**.** In $U = \mathbb{D} \setminus [0, 1)$, consider a point $a \in (0, 1)$. Explain why the boundary is locally a Jordan arc, why the exterior-cone hypothesis fails there, and how the two-sided construction repairs it.

Solution. A sufficiently small disk around a meets the removed radial segment in a straight diameter. Points of U occur on both sides of this diameter, so no open cone with positive angle can be placed in the local complement. Thus Proposition 13.1 does not apply to the whole local domain. The upper and lower half-disks are Jordan domains. Solve separate Poisson problems on them with the same boundary data $\rho(z) = |z - a|^2$. Their positive harmonic solutions agree with ρ on the shared diameter and have their unique zero there at a. Pasting their continuous boundary extensions gives the strict local barrier of Theorem 13.3. The slit tip 0 requires a separate local geometry argument; it was not an interior point of the arc in this example.

Chapter 14

The continuous Dirichlet operator

Source correspondence. Lyubich, Corollary 10.20 and the Banach-space statement of Section 10.8.

Assume now that every point of ∂U has a strict local barrier. The Perron envelope then acquires its entire prescribed boundary trace. Uniqueness turns existence into an operator, and the maximum principle turns that operator into an exact norm-preserving correspondence. This structure supports stable approximation of data without assuming differentiability of the boundary function.

14.1 Existence, uniqueness and the boundary isometry

Theorem 14.1 (Continuous Dirichlet problem on a regular domain)**.** *For every $g \in C(\partial U)$ there is a unique $E_U g \in C(\overline{U})$ harmonic on U with $(E_U g)|_{\partial U} = g$. It is the Perron envelope H_g in the interior. Moreover,*

$$\|E_U g\|_{C(\overline{U})} = \|g\|_{C(\partial U)}, \qquad g \geq 0 \Longrightarrow E_U g \geq 0, \qquad E_U 1 = 1.$$

Proof. Proposition 12.1 gives a bounded harmonic H_g. Theorem 12.5 gives $H_g(z) \to g(a)$ at every boundary point a. Define $E_U g$ by H_g on U and g on the boundary. It is continuous in the interior. At a boundary point, continuity follows by combining the interior limit with continuity of g along ∂U: a sequence in $\overline{U}$ splits into interior and boundary subsequences, both with the required limit.

If h_1, h_2 are two solutions, their harmonic difference is continuous on the compact closure and zero on the boundary. The boundary maximum and minimum principles give $h_1 - h_2 = 0$. For the norm, the same principles give $|E_U g| \leq \|g\|_\infty$ on U, and the boundary values attain that norm on the compact boundary, so equality holds on $\overline{U}$. Positivity follows by comparison with zero; the constant one is already a harmonic solution of its own boundary data and hence equals $E_U 1$. □

Proposition 14.2 (Linearity and completeness)**.** *Let*

$$\mathcal{H}_c(U) = \{h \in C(\overline{U}) : h|_U \text{ harmonic}\}.$$

Restriction to the boundary and E_U are inverse linear isometries between $\mathcal{H}_c(U)$ and $C(\partial U)$, both with their uniform norms. In particular $\mathcal{H}_c(U)$ is a Banach space.

Proof. For real a, b and continuous g, k, the function $aE_U g + bE_U k$ is harmonic and has boundary trace $ag + bk$. Uniqueness gives

$$E_U(ag + bk) = aE_U g + bE_U k.$$

This argument handles negative coefficients without claiming that the subharmonic competitor class itself is closed under negative scaling. Restriction followed by extension fixes every element of $\mathcal{H}_c(U)$ by uniqueness; extension followed by restriction fixes every boundary datum by construction. The norm identity is Theorem 14.1.

If h_n is Cauchy in $\mathcal{H}_c(U)$, its traces g_n are Cauchy in the complete space $C(\partial U)$. Let $g_n \to g$ uniformly. Then

$$\|h_n - E_U g\|_{C(\overline{U})} = \|g_n - g\|_{C(\partial U)} \longrightarrow 0.$$

Thus the limit lies in $\mathcal{H}_c(U)$ and the space is complete. Alternatively, uniform convergence on the closure gives local uniform convergence in U, which preserves harmonicity through mean values; the isometric argument additionally identifies the limit's trace. □

14.2 Comparison, approximation and a converse to regularity

Corollary 14.3 (Exact stability under boundary perturbation)**.** *For $g, k \in C(\partial U)$,*

$$g \le k \Longrightarrow E_U g \le E_U k, \qquad \|E_U g - E_U k\|_{C(\overline{U})} = \|g - k\|_{C(\partial U)}.$$

Consequently uniform approximation of boundary data gives uniform approximation of the solutions on the whole closed domain.

Proof. Apply positivity and the norm identity to $E_U(k - g) = E_U k - E_U g$. The equality includes the boundary; on a fixed compact interior subset the norm may be strictly smaller. If $\|g_n - g\|_\infty \le \varepsilon_n$, then the exact operator bound gives $\|E_U g_n - E_U g\|_\infty \le \varepsilon_n$. There is no accumulated iteration factor because the operator norm is one. □

Proposition 14.4 (Interior derivatives of a boundary error)**.** *Let $g, k \in C(\partial U)$ and put $\delta = \|g - k\|_{C(\partial U)}$. Suppose a coordinate disk $\overline{D(a, R)}$ lies in U. For every real multi-index $\alpha = (\alpha_1, \alpha_2)$ of length $m \ge 1$ and $0 \le \rho < R$,*

$$\sup_{|z-a| \le \rho} \left| \partial_x^{\alpha_1} \partial_y^{\alpha_2} (E_U g - E_U k)(z) \right| \le \frac{2m!\,\delta}{(R - \rho)^m}.$$

These derivatives are in the specified coordinate, not invariant norms on an unspecified metric surface.

Proof. Write $u = E_U g - E_U k$, so $|u| \le \delta$ everywhere. On a disk $D(z_0, s)$ whose closed disk lies in U, take a holomorphic function F with real part u and imaginary part zero at z_0. Its Taylor expansion and the Poisson Fourier coefficients give

$$F(z_0 + w) = u(z_0) + \sum_{j \ge 1} c_j w^j, \qquad c_j = \frac{2}{s^j} \int_0^{2\pi} u(z_0 + s e^{it}) e^{-ijt} \frac{\mathrm{d}t}{2\pi}.$$

Hence $|c_j| \le 2\delta / s^j$. Since $\partial_x^{\alpha_1} \partial_y^{\alpha_2} u(z_0) = \mathrm{Re}(i^{\alpha_2} m! c_m)$, its absolute value is at most $2m!\delta / s^m$. For $|z_0 - a| \le \rho$, any $s < R - \rho$ is allowed. Let $s \uparrow R - \rho$ and then take the supremum over z_0. The strict radius inequality justifies every Fourier calculation before the limiting estimate is taken. □

Thus uniform boundary approximation controls all derivatives on every fixed compact interior disk, even for boundary functions which are merely continuous. The blow-up factor $(R - \rho)^{-m}$ explains why such a statement does not supply uniform derivative control all the way to a rough boundary. For $u_n(z) = \mathrm{Re}(z^n)$ on $\mathbb{D}$, the boundary norm is one but $|\nabla u_n(1)| = n$; the interior bound on $|z| \le \rho < 1$ instead includes the damping factor $n\rho^{n-1}$. The norms and the region of control must both remain in the statement.

Proposition 14.5 (Solvability implies a barrier)**.** *Suppose $U \Subset S$ is connected, $|\partial U| \geq 2$, and every continuous boundary datum admits a continuous harmonic extension. Then every point of ∂U possesses a strict global barrier.*

Proof. Fix a metric d compatible with the topology on a neighborhood of the compact set $\overline{U}$. Such a metric is available on a Riemann surface. For $a \in \partial U$ prescribe

$$g_a(\xi) = d(\xi, a)^2 \quad (\xi \in \partial U).$$

Let h_a be its continuous harmonic extension. It is nonnegative by the maximum principle and has boundary zero only at a. If the boundary has another point, then g_a is not identically zero, so the strong minimum principle makes $h_a > 0$ throughout U. Hence h_a is a strict global barrier.

The hypothesis $|\partial U| \geq 2$ was used exactly to ensure that the distance-squared boundary datum is not identically zero. No smoothness of the boundary and no conformal parametrization were required. □

Remark 14.6 (The singleton-boundary qualification)*.* The hypothesis $|\partial U| \geq 2$ in the preceding proposition is necessary. For example, a compact connected Riemann surface with one point removed is relatively compact in the original surface and has singleton boundary. Every function on that boundary is constant and extends harmonically. In fact no strict positive superharmonic barrier exists at the puncture. If b were such a barrier, choose a coordinate disk around a on which U is the punctured disk. Then $-b$ is subharmonic, bounded above by zero, and tends to zero at a. The subharmonic removal theorem, Theorem 7.6, extends it across a with value zero. This extension attains its global maximum zero at the center and is strictly negative elsewhere, contradicting the strong maximum principle in Theorem 6.3. When a boundary has at least two points, the distance-squared construction provides the claimed equivalence between regularity and full continuous Dirichlet solvability. The forward implication needs no such extra condition.

Proposition 14.7 (Classification of the singleton exception)**.** *Let S be a connected Riemann surface without boundary, and let $U \Subset S$ be a nonempty connected open set with $\partial U = \{a\}$. Then S is compact and $U = S \setminus \{a\}$. Conversely, removing one point from a compact connected Riemann surface has these properties. In this case every continuous-on-closure harmonic function is constant.*

Proof. First $S \setminus \{a\}$ is connected. To verify the needed elementary topological fact, join two of its points by a path in S. A surface is locally path connected, so connectedness of S gives such a path. Choose a coordinate disk around a with both endpoints outside its closure. If the path meets a smaller concentric closed disk, use its first and last intersection with the boundary circle and replace the intervening segment by a circular arc. Compactness of the parameter interval supplies the first and last times, and continuity puts their images on the circle. The unchanged initial and final segments stay outside that smaller disk; the replacement lies on its circle. The resulting path avoids a.

Now U is relatively open in $S \setminus \{a\}$ and also relatively closed there, since

$$\overline{U}^{\,S} \setminus U = \partial U = \{a\}, \qquad \overline{U}^{\,S\setminus\{a\}} = U.$$

It is nonempty, so connectedness gives $U = S \setminus \{a\}$. The closure of U in S is all of S; relative compactness therefore makes S compact. Conversely the same path argument gives connectedness after removing one point from a compact surface, and its boundary is precisely that point.

If $h \in C(S)$ is harmonic on $S \setminus \{a\}$, continuity makes it bounded near a. Harmonic removability extends it harmonically across a, with the same value because of continuity. It then attains a maximum on the compact surface, so the strong maximum principle makes it constant. □

This identifies the exact trace-space degeneracy: both the harmonic space and $C(\{a\})$ consist of constants, and restriction is an isometry between one-dimensional spaces. The evaluation functional is represented by the point mass δ_a at every interior point; its weak concentration at a is then tautological and gives no strict barrier. In a noncompact connected ambient surface, in particular in the plane, the singleton configuration above cannot occur for a relatively compact nonempty domain. Thus the qualification in the converse is essential on compact surfaces without creating an extra exception for bounded planar domains.

14.3 Exact annular data and local consistency

For the annulus $A_\rho = \{\rho < |z| < 1\}$, $0 < \rho < 1$, the harmonic function with value one on the inner circle and zero on the outer is

$$w_\rho(z) = \frac{\log|z|}{\log\rho}.$$

Indeed, $\Delta \log|z| = 0$ away from zero, the boundary values are correct, and uniqueness identifies this formula with the Dirichlet solution. For constant data a on the inner circle and b on the outer,

$$E_{A_\rho} g = b + (a-b)\frac{\log|z|}{\log\rho}.$$

The dependence on logarithmic radius is affine, even though the dependence on Euclidean radius is not.

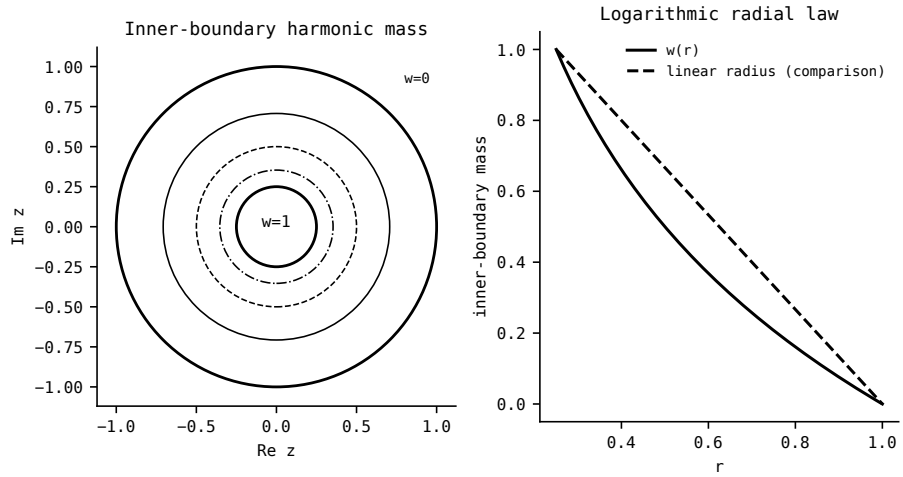


Figure 14.1: The exact inner-boundary mass $w_\rho(r) = \log r/\log\rho$ for $\rho = 1/4$. Concentric equipotentials and the radial profile display the same solution. The dashed comparison interpolates linearly in Euclidean radius; the harmonic profile instead interpolates linearly in logarithmic radius.

Proposition 14.8 (Consistency on an interior subdomain)**.** *Let $V \Subset U$ be a regular domain and $g \in C(\partial U)$. Then*

$$(E_U g)|_V = E_V\big((E_U g)|_{\partial V}\big).$$

Proof. The left side is harmonic in V and continuous on its closure because the closure lies inside U. It has exactly the displayed boundary data. Uniqueness on V identifies the two functions. The statement is not that one may arbitrarily replace outer boundary data by values at sampled interior points; the whole trace on ∂V is required. □

14.4 Uniform families of boundary data

The operator norm one gives stability for a single convergent sequence. Compactness of a family requires an additional distinction between interior and boundary behavior. A uniformly bounded collection of boundary data gives locally normal harmonic extensions by interior derivative estimates, but it need not give a compact family on the entire closed domain: the traces might oscillate arbitrarily rapidly.

Proposition 14.9 (Compactness up to a regular boundary)**.** *Let $\mathcal{G} \subset C(\partial U)$ be uniformly bounded and equicontinuous. Then $E_U\mathcal{G}$ is relatively compact in $C(\overline{U})$. Every uniform limit in its closure is harmonic in U.*

Proof. The boundary is compact and metrizable. Arzelà–Ascoli gives relative compactness of $\mathcal{G}$ in $C(\partial U)$. Given a sequence $g_n \in \mathcal{G}$, choose a subsequence g_{n_j} converging uniformly to some $g \in C(\partial U)$. Exact stability yields

$$\|E_U g_{n_j} - E_U g\|_{C(\overline{U})} = \|g_{n_j} - g\|_{C(\partial U)} \longrightarrow 0.$$

Thus every sequence of extensions has a uniformly convergent subsequence with harmonic limit. This sequential characterization is equivalent to relative compactness in the metric space $C(\overline{U})$.

One can also see the boundary control directly from barriers. Let $|g| \le M$ for all $g \in \mathcal{G}$. Equicontinuity gives, for each $\varepsilon > 0$, a neighborhood of each boundary point on which all data vary by at most ε. The comparison in Theorem 12.5 then uses a coefficient $(2M + \varepsilon)/c_\varepsilon$ independent of g. Since a fixed barrier vanishes at its distinguished point, all extensions approach their traces uniformly over the family there. A finite boundary cover and interior harmonic equicontinuity complete a direct Arzelà–Ascoli proof on the closure. The operator argument is shorter; the barrier argument explains the geometry behind the same compactness. □

For comparison, take $g_n(e^{it}) = \cos(nt)$ on the unit circle. The data are bounded by one, and their extensions are

$$h_n(re^{it}) = r^n \cos(nt).$$

For every $0 < r_0 < 1$,

$$\sup_{|z| \le r_0} |h_n(z)| \le r_0^n \longrightarrow 0, \qquad \|h_n\|_{C(\overline{\mathbb{D}})} = 1.$$

There is no subsequence converging uniformly on the closed disk to zero, and any hypothetical uniformly convergent subsequence would have to have that same zero limit in the interior and hence by continuity on the closure. Therefore no uniformly convergent subsequence exists on the closure. Interior normality and compactness up to the boundary are distinct results.

A second useful consequence of the isometry concerns boundary approximation spaces. Let $V_n \subset C(\partial U)$ be any finite-dimensional subspace, without assuming that it comes from a numerical grid. Then

$$\inf_{v \in V_n} \|E_U g - E_U v\|_{C(\overline{U})} = \inf_{v \in V_n} \|g - v\|_{C(\partial U)}.$$

Thus best uniform approximation of the exact solution by exact harmonic extensions of V_n is precisely the boundary best-approximation problem. If the proposed interior approximants are not exact harmonic extensions, an additional interior residual or comparison analysis is needed; the identity does not apply to arbitrary arrays of nodal values.

14.5 Five worked investigations

Investigation 14.10 (Fourier data on a disk)**.** Find the harmonic extension of $g(e^{it}) = 3 + 2\cos(2t) - \sin(3t)$ and bound the extension error after dropping the last term on $|z| \le r$.

Solution. Harmonicity of real and imaginary parts of powers gives

$$E_{\mathbb{D}}g(re^{it}) = 3 + 2r^2\cos(2t) - r^3\sin(3t).$$

The expression is continuous on the closed disk and has the correct trace, so uniqueness verifies it. Dropping the last term gives error at most r^3 on the radius-r disk, with equality for suitable angles on its boundary. The global closed-disk error is one, matching the boundary norm. Thus the isometry on the whole closure coexists with stronger interior smoothing for oscillatory boundary modes.

Investigation 14.11 (An annular interpolation value)**.** For $\rho = 1/4$, inner value 5 and outer value -1, compute the solution at radius $1/2$ and its radial derivative.

Solution. The formula is

$$h(r) = -1 + 6\frac{\log r}{\log(1/4)}, \qquad h(1/2) = -1 + 6\cdot\frac{1}{2} = 2, \qquad h'(r) = \frac{6}{r\log(1/4)} < 0.$$

The midpoint in logarithmic radius gives the arithmetic midpoint of the boundary values. At the Euclidean midpoint $r = 5/8$, the interpolation weight is $\log(5/8)/\log(1/4)$, which is not $1/2$. This distinction is the continuous analogue of the logarithmic radial resistance of a two-dimensional annulus.

Investigation 14.12 (A uniform limit without differentiated convergence)**.** Suppose $g_n \to g$ uniformly on ∂U. Explain why the corresponding harmonic solutions converge even if no derivatives of g_n exist.

Solution. The norm identity gives

$$\sup_{\overline{U}} |E_U g_n - E_U g| = \sup_{\partial U} |g_n - g| \longrightarrow 0.$$

Only continuity of the boundary functions was assumed. Interior derivatives can subsequently be controlled on compact subsets by differentiated Poisson kernels, but those estimates are consequences of interior harmonicity, not requirements on the boundary data. In particular, piecewise linear continuous approximations to a rough continuous boundary function can converge uniformly while their slopes oscillate or become unbounded; the uniform solution convergence remains valid.

Investigation 14.13 (Why subtraction is postponed until uniqueness)**.** The Perron construction uses subharmonic competitors. Why is $E_U(-g) = -E_U g$ valid although negatives of subharmonic competitors need not be subharmonic?

Solution. Once $E_U g$ is proved harmonic and continuous on the closure, its negative is harmonic and has boundary trace $-g$. The uniqueness theorem therefore gives the identity. There is no assertion that $\mathcal{P}_U(-g) = -\mathcal{P}_U(g)$; the latter set generally consists of superharmonic functions. The linear operator structure emerges after the Perron envelope becomes a harmonic solution. Applying algebraic identities to the competitor classes before that step would confuse lower and upper comparison families.

Investigation 14.14 (The sole boundary point)**.** On $S\backslash\{a\}$ with S a compact connected Riemann surface, explain why constant boundary data have unique continuous harmonic extensions but do not produce a strict barrier.

Solution. Every datum on the singleton $\{a\}$ is a constant c, and $h \equiv c$ is an extension. A second continuous extension differs by a harmonic function tending to zero at a; the maximum

principle on compact closure gives zero. Yet the nonnegative distance-squared datum on that singleton is identically zero, so its extension is zero throughout the domain. It cannot separate a from interior points. Indeed, if b were a strict local positive superharmonic barrier, then $-b$ would extend subharmonically across the puncture with value zero by Theorem 7.6. Its global maximum at that interior point would force it to be constant by Theorem 6.3, contradicting strict negativity away from the puncture. This is why a converse based solely on all continuous boundary data must exclude singleton boundaries, even though the forward regular-domain theorem is unchanged.

Chapter 15

Harmonic measure as an exact boundary functional

Source correspondence. Lyubich, Section 10.8 and the normalization in Proposition 10.4.

The Dirichlet operator converts a boundary function into a harmonic function. Evaluating the result at an interior point converts it into a number. Since that evaluation is positive, linear and normalized, it can be represented by a probability measure on the boundary. Harmonic measure is this measure; it records how each boundary region contributes to the value at a specified observation point. The measure generally changes with that point.

We continue to assume that $U \Subset S$ is connected, has nonempty boundary, and that every boundary point has a strict local barrier. No stochastic process is needed for the definition or the proofs below. An interpretation as a Brownian exit distribution requires additional probability-theoretic construction and is not asserted as a simulated or measured result here.

15.1 The representation theorem and its exact inputs

We use the following classical form of the Riesz–Markov representation theorem as an explicit input: if K is compact metrizable and $L : C(K, \mathbb{R}) \to \mathbb{R}$ is a positive linear functional, there exists a unique finite regular Borel measure μ such that

$$L(f) = \int_K f \, \mathrm{d}\mu \qquad (f \in C(K)), \qquad \mu(K) = L(1).$$

Positivity means $f \geq 0 \Rightarrow L(f) \geq 0$. Regularity means that Borel sets can be approximated in measure from inside by compact sets and from outside by open sets. This functional-representation theorem is a named measure-theoretic dependency; it is not proved by the harmonic arguments.

Theorem 15.1 (Harmonic measure)**.** *For each $z \in U$, there is a unique regular Borel probability measure ω_U^z on ∂U such that*

$$(E_U g)(z) = \int_{\partial U} g(\xi) \, \mathrm{d}\omega_U^z(\xi) \qquad (g \in C(\partial U)).$$

Proof. Define $L_z(g) = (E_U g)(z)$. Linearity of E_U makes L_z linear; positivity of E_U makes it positive. Furthermore,

$$|L_z(g)| \leq \|E_U g\|_\infty = \|g\|_\infty, \qquad L_z(1) = 1.$$

Thus L_z is bounded with norm exactly one. The compact boundary of a Riemann surface is metrizable, so the stated representation theorem gives a unique finite regular Borel measure.

Its total mass is $L_z(1) = 1$, proving that it is a probability measure. Every property invoked here has been identified: the analytic input is the Dirichlet operator, and the measure input is positive-functional representation. □

For the unit disk, comparison with the Poisson formula identifies the measure explicitly:

$$\mathrm{d}\omega_{\mathbb{D}}^{z}(e^{it}) = \frac{1-|z|^2}{|e^{it}-z|^2}\frac{\mathrm{d}t}{2\pi}.$$

At the center this is normalized angular measure. Omitting the factor $1/(2\pi)$ would assign mass 2π to the boundary and would fail to reproduce the constant function one. The denominator can instead be absorbed into the definition of normalized angular measure, but the chosen convention must remain consistent in every formula.

15.2 Dependence on the interior point and the boundary limit

Proposition 15.2 (Weak continuity and concentration)**.** *The map $z \mapsto \omega_U^z$ is weakly continuous on U. At every boundary point a,*

$$\omega_U^z \longrightarrow \delta_a \quad \text{weakly as } U \ni z \to a.$$

Every nonempty relatively open subset of ∂U has positive harmonic measure at every interior point.

Proof. Weak continuity means continuity of the integral against each continuous boundary function. But

$$\int g\,\mathrm{d}\omega_U^z = (E_U g)(z)$$

is harmonic, hence continuous, in z. The boundary limit is equally direct: continuity of the solution on $\overline{U}$ gives $\int g\,\mathrm{d}\omega_U^z \to g(a) = \int g\,\mathrm{d}\delta_a$.

Let $O \subset \partial U$ be nonempty and relatively open. Choose $a \in O$ and a continuous function f with $0 \le f \le 1$, $f(a) > 0$ and support in O. For a compact metric space one can construct such a function from distances to a and to $\partial U \setminus O$; if $O = \partial U$, use $f = 1$. The harmonic extension $E_U f$ is nonnegative and not identically zero, because its boundary trace is not zero. The strong minimum principle therefore gives $(E_U f)(z) > 0$ throughout the connected domain. Finally,

$$0 < (E_U f)(z) = \int f\,\mathrm{d}\omega_U^z \le \omega_U^z(O).$$

This proves full topological support; it does not say that every singleton has positive measure. □

Proposition 15.3 (Harnack comparison of boundary measures)**.** *Fix $z_0 \in U$ and a compact set $K \Subset U$. There is $C_K \ge 1$ such that for every Borel set $A \subset \partial U$ and every $z \in K$,*

$$C_K^{-1}\omega_U^{z_0}(A) \le \omega_U^z(A) \le C_K\omega_U^{z_0}(A).$$

In particular all interior harmonic measures have the same null sets.

Proof. For any continuous $g \ge 0$, its harmonic extension is nonnegative. Apply Harnack to $E_U g + \varepsilon$ and let $\varepsilon \downarrow 0$ to include the zero function. The Harnack constant depends only on K, z_0, U, not on g, and gives

$$C_K^{-1}\int g\,\mathrm{d}\omega_U^{z_0} \le \int g\,\mathrm{d}\omega_U^{z} \le C_K\int g\,\mathrm{d}\omega_U^{z_0}.$$

To pass from continuous functions to sets, use regularity of the measures. For a compact $F \subset \partial U$, continuous functions decreasing to 1_F can be chosen as $g_n(\xi) = \max\{0, 1 - n\,\mathrm{dist}(\xi, F)\}$. Dominated convergence gives the inequality for F. Inner regularity then gives it for every Borel A by taking suprema over compact $F \subset A$. The lower inequality follows by the same argument. For a specified pair of points choose a compact set containing both; if one measure of A is zero, both displayed bounds force the other to vanish. □

This comparison is stronger than topological support. Two full-support probability measures can still have different null sets, whereas harmonic measures based at interior points are uniformly comparable on compact families of base points. The constants deteriorate as a point approaches the boundary, consistently with weak concentration at a Dirac mass.

15.3 Borel boundary sets and harmonic mass functions

Proposition 15.4 (Harmonicity for a fixed Borel set)**.** *For every Borel set $A \subset \partial U$, the function*

$$z \longmapsto \omega_U^z(A)$$

is harmonic on U and takes values in $[0, 1]$. If A is clopen in ∂U, it extends continuously to the boundary with value one on A and zero on its complement.

Proof. First let O be relatively open. If $O \neq \partial U$, the functions

$$f_n(\xi) = \min\{1, n\,\mathrm{dist}(\xi, \partial U \setminus O)\}$$

are continuous, increase pointwise to 1_O, and lie in $[0, 1]$. Their harmonic extensions increase, are bounded above by one, and satisfy

$$(E_U f_n)(z) = \int f_n \,\mathrm{d}\omega_U^z \longrightarrow \omega_U^z(O)$$

by monotone convergence. The harmonic monotone convergence theorem makes the limit harmonic. The case $O = \partial U$ is the constant one.

To reach all Borel sets, let $\mathcal{L}$ be the collection of sets whose mass function is harmonic. It contains the whole boundary and every open set. If $A \in \mathcal{L}$, then $\omega_U^z(A^c) = 1 - \omega_U^z(A)$ is harmonic. If the A_j are pairwise disjoint members, their finite-sum mass functions are harmonic, increase and are bounded by one. Monotone harmonic convergence again shows that $\omega_U^z(\bigcup_j A_j) = \sum_j \omega_U^z(A_j)$ is harmonic. Thus $\mathcal{L}$ is a Dynkin system containing the open sets. Open sets form a collection closed under finite intersection and generate the Borel sigma algebra. The elementary pi–lambda theorem therefore gives all Borel sets. This set-theoretic theorem is used only to extend the proved closure properties, not to insert a new analytic assumption.

The set-theoretic step can be verified directly. Write $K = \partial U$, let $\mathcal{P}$ be the relatively open subsets, and let $\mathcal{D}$ be the smallest Dynkin system containing $\mathcal{P}$. A Dynkin system is closed under differences $B \setminus A$ when $A \subset B$ are both members: decompose $A^c = (B \setminus A) \sqcup B^c$ and use complements and disjoint unions, or equivalently write $B \setminus A = (A \cup B^c)^c$. For fixed $A \in \mathcal{P}$, the class

$$\mathcal{D}_A = \{B \subset K : A \cap B \in \mathcal{D}\}$$

is a Dynkin system. Its whole-space condition uses $A \in \mathcal{D}$; its complement condition uses $A \setminus (A \cap B)$; its disjoint-union condition follows by intersecting each disjoint member with A. Because open sets are closed under intersection, $\mathcal{P} \subset \mathcal{D}_A$, and minimality gives $\mathcal{D} \subset \mathcal{D}_A$. Now fix $B \in \mathcal{D}$ and repeat the argument for $\{A : A \cap B \in \mathcal{D}\}$. The preceding conclusion makes

this Dynkin system contain $\mathcal{P}$, so it contains $\mathcal{D}$. Thus $\mathcal{D}$ is closed under finite intersections. Complements give finite unions, and any countable union can be written as a disjoint union

$$\bigcup_{j\geq 1} A_j = \bigsqcup_{j\geq 1} \left(A_j \setminus \bigcup_{k<j} A_k \right).$$

Hence $\mathcal{D}$ is a sigma algebra and contains the Borel sigma algebra generated by the open sets. Since $\mathcal{L}$ contains $\mathcal{D}$, every Borel mass function is harmonic. This spells out exactly why the analytic closure under disjoint increasing sums suffices; closure under an arbitrary pointwise supremum of harmonic functions was never asserted.

If A is clopen, its indicator 1_A is continuous. The mass function is then $E_U 1_A$, whose continuous boundary trace is exactly 1_A. For a general Borel set, its indicator need not be continuous, and no continuous trace at points of its relative boundary has been claimed. □

Corollary 15.5 (A dichotomy for a boundary set)**.** *For a fixed Borel A, either $\omega_U^z(A) = 0$ at every interior point, or it is strictly positive at every interior point. Likewise it is either identically one or strictly less than one everywhere.*

Proof. Apply the strong minimum principle to the nonnegative harmonic mass function, and then to its complementary mass $1 - \omega_U^z(A)$. This also follows from Harnack comparison. For a nonempty proper clopen set, both it and its complement are nonempty open sets, so the mass is strictly between zero and one throughout U. □

15.4 Conformal transport and localized perturbations

Proposition 15.6 (Conformal covariance)**.** *Let $F : U \to V$ be conformal and extend to a homeomorphism $\overline{U} \to \overline{V}$, where both domains satisfy the regular-domain hypotheses. Then*

$$F_* \omega_U^z = \omega_V^{F(z)}, \qquad \omega_U^z(F^{-1}A) = \omega_V^{F(z)}(A)$$

for every Borel $A \subset \partial V$.

Proof. For $g \in C(\partial V)$, the function $(E_V g) \circ F$ is harmonic in U by conformal invariance and has boundary trace $g \circ F$. Uniqueness gives

$$E_U(g \circ F) = (E_V g) \circ F.$$

Evaluating at z and using the measure representations yields

$$\int_{\partial V} g \, \mathrm{d}(F_* \omega_U^z) = \int_{\partial U} g \circ F \, \mathrm{d}\omega_U^z = (E_V g)(F(z)) = \int_{\partial V} g \, \mathrm{d}\omega_V^{F(z)}.$$

Uniqueness of regular Borel measures with the same continuous integrals gives the pushforward identity and hence the Borel-set formula. The extension hypothesis matters: a conformal equivalence without an ordinary boundary homeomorphism requires a prime-end formulation instead. □

Proposition 15.7 (Boundary error weighted by harmonic measure)**.** *Let $g, k \in C(\partial U)$ and let A be Borel. Suppose $|g - k| \leq M$ on A and $|g - k| \leq \delta$ on its complement, with $0 \leq \delta \leq M$. Then*

$$|(E_U g)(z) - (E_U k)(z)| \leq \delta + (M - \delta)\omega_U^z(A).$$

Proof. Linearity, the triangle inequality for integrals, and the stated pointwise bounds give

$$\begin{aligned}|E_U(g-k)(z)| &\le \int_{\partial U} |g-k| \,\mathrm{d}\omega_U^z \\ &\le M\omega_U^z(A) + \delta\omega_U^z(A^c) = \delta + (M-\delta)\omega_U^z(A).\end{aligned}$$

No area, arclength or sampling density appears in this bound. The relevant size of the exceptional boundary region is its harmonic measure from the observation point. A geometrically small set can carry a large mass near that part of the boundary. □

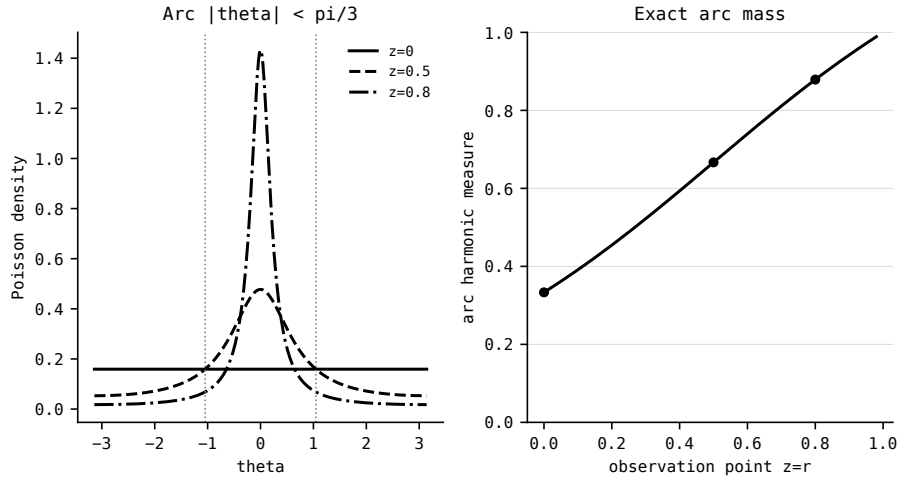


Figure 15.1: The boundary arc $|\theta| < \pi/3$ and normalized Poisson densities for real base points 0, 1/2 and 4/5. The arc mass at $z = r$ is $\frac{2}{\pi}\arctan\big(\frac{1+r}{1-r}\tan(\pi/6)\big)$. The density concentrates near $\theta = 0$ as the observation point approaches that boundary point; the total mass remains one.

15.5 Nested-domain averaging and convex inequalities

Harmonic measure is compatible with solving the problem in stages. This is an exact identity for nested regular domains; it needs no interpretation as an unconstructed random stopping process.

Proposition 15.8 (Composition through an interior boundary)**.** *Let $V \Subset U$ be a regular domain. For every Borel set $A \subset \partial U$ and every $z \in V$,*

$$\omega_U^z(A) = \int_{\partial V} \omega_U^\xi(A) \,\mathrm{d}\omega_V^z(\xi).$$

Proof. By Proposition 15.4, $h_A(\xi) = \omega_U^\xi(A)$ is harmonic on U and bounded between zero and one. It is therefore continuous on a neighborhood of the compact set $\overline{V}$. Its restriction to V is the unique continuous Dirichlet solution with boundary values $h_A|_{\partial V}$. Applying the defining representation of harmonic measure on V gives

$$h_A(z) = E_V(h_A|_{\partial V})(z) = \int_{\partial V} h_A(\xi) \,\mathrm{d}\omega_V^z(\xi).$$

Substitution is exactly the claimed formula. The compact inclusion makes every point of ∂V an interior point of U, so no unspecified boundary value of the Borel mass function is used. □

For $U = \mathbb{D}$, $V = D(0, r)$ and $z = 0$, this identity says that the centered harmonic measure on the unit circle is the average of the measures based at the intermediate circle:

$$\omega_{\mathbb{D}}^{0}(A) = \frac{1}{2\pi} \int_{0}^{2\pi} \omega_{\mathbb{D}}^{re^{it}}(A)\, \mathrm{d}t.$$

The individual Poisson kernels are highly nonuniform when r is near one, but their rotational average is uniform. This follows either from the composition identity or directly from the mean value property of $z \mapsto P(z, \zeta)$ for each fixed ζ.

15.5.1 Kernel identities for every Borel datum

The set identity extends to integration without an unmentioned interchange of limits. For a nonnegative simple Borel function $f = \sum_{j=1}^{m} c_j 1_{A_j}$, $c_j \geq 0$, multiply the composition formula for A_j by c_j and add. The result is

$$\int_{\partial U} f\, \mathrm{d}\omega_U^z = \int_{\partial V} \left(\int_{\partial U} f(\eta)\, \mathrm{d}\omega_U^{\xi}(\eta) \right) \mathrm{d}\omega_V^z(\xi). \tag{15.1}$$

For a general nonnegative Borel f, choose nonnegative simple functions $f_n \uparrow f$. Each inner integral is a finite sum of continuous Borel mass functions of ξ; its increasing limit is therefore measurable. Monotone convergence first in the inner integral and then in the outer integral proves (15.1), allowing $+\infty$ on both sides. For signed f, apply this argument to f^+ and f^-; subtraction is legitimate whenever the same formula for $|f|$ is finite. In particular every bounded Borel datum is covered.

For three nested regular domains $W \Subset V \Subset U$, apply the same identity twice. With $z \in W$ and nonnegative Borel f on ∂U,

$$\int_{\partial U} f\, \mathrm{d}\omega_U^z = \int_{\partial W} \left[\int_{\partial V} \left(\int_{\partial U} f(\eta)\, \mathrm{d}\omega_U^{\xi}(\eta) \right) \mathrm{d}\omega_V^{\zeta}(\xi) \right] \mathrm{d}\omega_W^z(\zeta).$$

Each intermediate boundary is compactly contained in the next domain; every evaluation point therefore lies in a region where its mass functions are harmonic. This is associativity of the exact extension kernels, with all variables and integration spaces specified. It does not identify the value of a discontinuous datum at an irregular boundary point.

15.5.2 An off-center disk identity and its Fourier specialization

Let $V = D(c, R)$ with $|c| + R < 1$, and write $z = c + Rw$, $|w| < 1$. For every $\eta = e^{i\theta}$ on the unit circle, the Poisson kernel $\xi \mapsto P(\xi, \eta)$ is harmonic on the unit disk and continuous near $\overline{V}$. Applying the disk formula to that specific harmonic function gives the pointwise identity

$$P(c + Rw, e^{i\theta}) = \int_{0}^{2\pi} P(c + Re^{it}, e^{i\theta}) P(w, e^{it}) \frac{\mathrm{d}t}{2\pi}. \tag{15.2}$$

Indeed, $P(\xi, \eta) = \mathrm{Re}((\eta+\xi)/(\eta-\xi))$, and its denominator never vanishes near $\overline{V}$. If $q = |c|+R < 1$, the intermediate kernel satisfies the uniform bound

$$0 < P(c + Re^{it}, e^{i\theta}) \leq \frac{1+q}{1-q} \qquad (t, \theta \in \mathbb{R}).$$

Thus integrating the identity against any bounded Borel datum causes no integrability problem. With ordinary, unnormalized angles, the joint iterated density is

$$I(t, \theta) = \frac{P(w, e^{it}) P(c + Re^{it}, e^{i\theta})}{(2\pi)^2}, \quad \int_{0}^{2\pi} I(t, \theta)\, \mathrm{d}t = \frac{P(c + Rw, e^{i\theta})}{2\pi}.$$

The other marginal is $P(w, e^{it})/(2\pi)$, since the outer-boundary kernel integrates to one in θ. Both marginals and total mass one follow analytically, independently of any discretized picture of I.

For concentric disks there is a second proof by Fourier multiplication. Put $p_r(t) = (1 - r^2)/(1 - 2r\cos t + r^2)$, $0 \le r < 1$. The geometric series yields an absolutely uniformly convergent expansion

$$p_r(t) = \sum_{n\in\mathbb{Z}} r^{|n|} e^{int}, \qquad \sum_{n\in\mathbb{Z}} r^{|n|} = \frac{1+r}{1-r}.$$

For $r, s < 1$, absolute summability permits termwise multiplication and integration. Orthogonality kills all terms with unequal Fourier indices:

$$\begin{aligned}\int_0^{2\pi} p_r(\theta - t) p_s(t - \varphi) \frac{\mathrm{d}t}{2\pi} &= \sum_{n\in\mathbb{Z}} r^{|n|} s^{|n|} e^{in(\theta-\varphi)} \\ &= p_{rs}(\theta - \varphi).\end{aligned}$$

Three stages similarly give $p_r * p_s * p_t = p_{rst}$ with normalized angular convolution. The radial damping factors multiply because the same Fourier mode is propagated through each concentric stage. This exact identity includes $r = 0$ or $s = 0$, where the resulting density is uniform, and excludes the singular boundary parameter $r = 1$ from the absolutely convergent series.

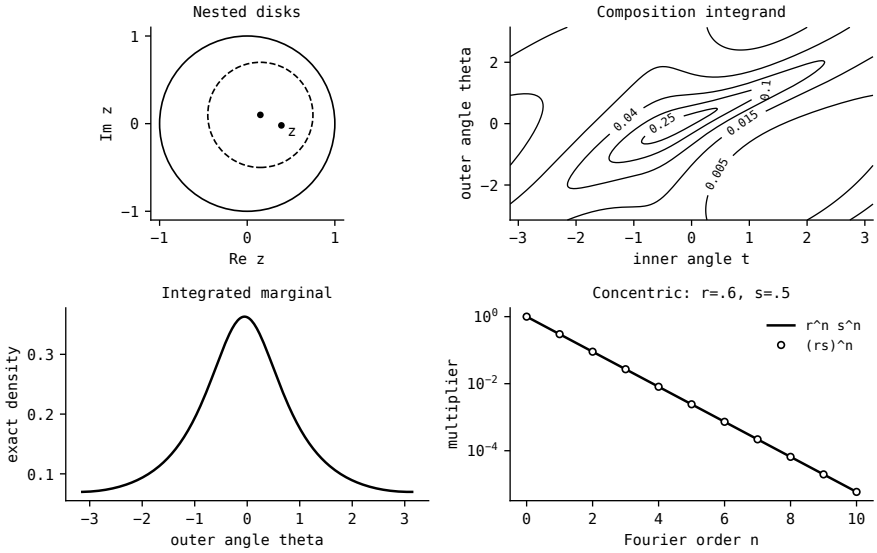


Figure 15.2: Composition through the off-center disk $D(c, R)$ with $c = 0.15 + 0.10i$, $R = 0.60$, $w = 0.40 - 0.20i$ and $z = c + Rw = 0.39 - 0.02i$. The geometry, iterated-density contours and exact marginal illustrate (15.2); the marginal is $P(z, e^{i\theta})/(2\pi)$. The concentric Fourier comparison records $r^{|n|}s^{|n|} = (rs)^{|n|}$. Plotted samples illustrate these analytic identities and do not replace their normalization or integration proofs.

Proposition 15.9 (Convex averaging and a variance function)**.** *Let $g \in C(\partial U)$ and let Φ be convex and continuous on an interval containing the range of g. Then*

$$\Phi(E_U g(z)) \le E_U(\Phi \circ g)(z).$$

In particular, with $h = E_U g$,

$$V_g = E_U(g^2) - h^2 \ge 0, \qquad V_g|_{\partial U} = 0, \qquad \Delta V_g = -2|\nabla h|^2 \le 0$$

in every conformal coordinate on U.

Proof. Write $m = \int g \, \mathrm{d}\omega_U^z$. Since ω_U^z is a probability, m lies between the minimum and maximum of g. A convex function on an interval has a supporting affine function at every interior point: $\Phi(t) \geq \Phi(m) + s(t-m)$ for a suitable slope s. This elementary fact follows by choosing s between the left and right secant-slope limits. Integrate the inequality to obtain

$$\int \Phi(g) \, \mathrm{d}\omega_U^z \geq \Phi(m) + s\left(\int g \, \mathrm{d}\omega_U^z - m\right) = \Phi(m).$$

If m is an endpoint of the range interval, the average of the nonnegative function $g - \min g$ (or $\max g - g$) is zero, so $g = m$ almost everywhere for this measure and the same conclusion follows with equality. This proves the convex inequality including endpoint cases.

For the square, one can avoid the supporting-line argument entirely:

$$E_U(g^2)(z) - h(z)^2 = \int_{\partial U} (g(\xi) - h(z))^2 \, \mathrm{d}\omega_U^z(\xi) \geq 0.$$

Both terms extend continuously to the boundary, where their difference is $g^2 - g^2 = 0$. Since h and $E_U(g^2)$ are harmonic and smooth in the interior, the product rule gives

$$\Delta(h^2) = 2h\Delta h + 2(h_x^2 + h_y^2) = 2|\nabla h|^2,$$

which proves the Laplacian formula. The numerical value of the gradient norm depends on the chosen coordinate metric, but its sign and the superharmonicity conclusion are conformally invariant. □

The variance is strictly positive in the interior whenever g is nonconstant. To see this, if the integral of $(g - h(z))^2$ vanished, its continuous nonnegative integrand would vanish on the full support of ω_U^z, namely all of ∂U. Then g would be constant. Thus the linear positive extension operator is usually not multiplicative:

$$E_U(g^2) \neq (E_U g)^2$$

for nonconstant data. On the disk with $g(e^{it}) = \cos t$, the exact example is

$$E_{\mathbb{D}} g = x, \qquad E_{\mathbb{D}}(g^2) = \frac{1 + x^2 - y^2}{2}, \qquad V_g(x + iy) = \frac{1 - x^2 - y^2}{2}.$$

The variance vanishes at the boundary, is positive inside, and has $\Delta V_g = -2$, exactly as the general formula predicts because $|\nabla x|^2 = 1$. This is an analytic identity, not an estimate inferred from a finite set of plotted values.

Proposition 15.10 (Variance splits across an intermediate boundary)**.** *Let $V \Subset U$ be regular, $g \in C(\partial U)$, and $h = E_U g$. With $V_g = E_U(g^2) - h^2$, every $z \in V$ satisfies*

$$\begin{aligned} V_g(z) = &\int_{\partial V} V_g(\xi) \, \mathrm{d}\omega_V^z(\xi) \\ &+ \int_{\partial V} (h(\xi) - h(z))^2 \, \mathrm{d}\omega_V^z(\xi). \end{aligned}$$

Both terms on the right are nonnegative. The first is the average residual variance after reaching the intermediate boundary; the second is the variance of the intermediate harmonic values themselves. These descriptions refer to the displayed integrals, without requiring a stochastic process.

Proof. The functions h and $E_U(g^2)$ are harmonic on U, hence continuous on a neighborhood of $\overline{V}$. Their restriction identities give

$$h(z) = \int_{\partial V} h(\xi) \, \mathrm{d}\omega_V^z(\xi), \quad E_U(g^2)(z) = \int_{\partial V} E_U(g^2)(\xi) \, \mathrm{d}\omega_V^z(\xi).$$

Add and subtract $\int_{\partial V} h(\xi)^2 \, \mathrm{d}\omega_V^z(\xi)$ in the difference $E_U(g^2)(z) - h(z)^2$. The first resulting term is the integral of V_g. In the second, expand the square:

$$\begin{aligned}\int_{\partial V} (h(\xi) - h(z))^2 \, \mathrm{d}\omega_V^z(\xi) &= \int_{\partial V} h(\xi)^2 \, \mathrm{d}\omega_V^z(\xi) - 2h(z) \int_{\partial V} h(\xi) \, \mathrm{d}\omega_V^z(\xi) \\ &\quad + h(z)^2 \\ &= \int_{\partial V} h(\xi)^2 \, \mathrm{d}\omega_V^z(\xi) - h(z)^2.\end{aligned}$$

The normalization of harmonic measure accounts for the final constant term. Substitution proves the identity. The first term is nonnegative because $V_g \geq 0$ on U, and the second is an integral of a square. □

For an exact check take $U = \mathbb{D}$, $V = D(0, R)$, $0 < R < 1$, and $g(e^{it}) = \cos t$. Then $h(z) = \operatorname{Re} z$ and $V_g(z) = (1 - |z|^2)/2$. On the intermediate circle the first integrand is the constant $(1 - R^2)/2$. The harmonic extension into $D(0, R)$ of $h(Re^{it})^2 = R^2 \cos^2 t$ equals

$$\frac{R^2 + \operatorname{Re}(z^2)}{2}.$$

After subtracting $h(z)^2 = (\operatorname{Re} z)^2$, the intermediate variance is $(R^2 - |z|^2)/2$. Thus the complete decomposition reads

$$\frac{1 - |z|^2}{2} = \frac{1 - R^2}{2} + \frac{R^2 - |z|^2}{2} \qquad (|z| < R).$$

This example checks the spatial dependence, not only the centered value. As R varies between $|z|$ and one, the two contributions redistribute while their sum remains fixed. The identity also shows directly that $V_g(z) \geq \int_{\partial V} V_g \, \mathrm{d}\omega_V^z$, an averaging inequality consistent with the previously computed nonpositive Laplacian of V_g.

15.6 Bounded Borel data and their continuity points

The measure representation defines an interior harmonic function even when the boundary datum is not continuous. The resulting boundary statement must be weakened carefully: one obtains the expected limit at every point where the datum is continuous, but not a continuous trace at an arbitrary jump.

Theorem 15.11 (Bounded measurable extension)**.** *For a bounded real Borel function f on ∂U, define*

$$T_U f(z) = \int_{\partial U} f(\xi) \, \mathrm{d}\omega_U^z(\xi).$$

Then $T_U f$ is harmonic on U, the map T_U is positive and linear, and

$$\|T_U f\|_{L^\infty(U)} \leq \|f\|_{L^\infty(\partial U)},$$

where the right side denotes the pointwise supremum norm. If f is continuous at $a \in \partial U$, then $T_U f(z) \to f(a)$ as $z \to a$ through U. For continuous f, $T_U f = E_U f$.

Proof. Choose bounded simple Borel functions s_n such that $\sup_{\partial U} |s_n - f| \leq 1/n$. One concrete choice rounds each value of f down to the nearest multiple of $1/n$; boundedness leaves only finitely many values, and the inverse-image sets are Borel. Each

$$s_n = \sum_{j=1}^{m_n} c_{n,j} 1_{A_{n,j}}$$

has harmonic extension $T_U s_n = \sum_j c_{n,j}\omega_U^z(A_{n,j})$ by Proposition 15.4. Probability normalization gives

$$|T_U s_n(z) - T_U f(z)| \le \int |s_n - f| \, \mathrm{d}\omega_U^z \le 1/n$$

uniformly on all of U. A locally uniform limit of harmonic functions is harmonic, so $T_U f$ is harmonic. Positivity, linearity and the norm bound follow directly from integration against a probability measure.

Fix a continuity point a and $\varepsilon > 0$. Choose a relative boundary neighborhood O of a on which $|f - f(a)| < \varepsilon$. There is a continuous cutoff χ with $0 \le \chi \le 1$, $\chi(a) = 1$ and support contained in O. Since $\chi \le 1_O$,

$$\omega_U^z(O^c) \le 1 - \int \chi \, \mathrm{d}\omega_U^z = 1 - (E_U\chi)(z) \longrightarrow 0$$

by boundary attainment for the continuous cutoff. Splitting the integral,

$$\begin{aligned} |T_U f(z) - f(a)| &\le \int |f - f(a)| \, \mathrm{d}\omega_U^z \\ &\le \varepsilon\, \omega_U^z(O) + 2\|f\|_\infty\, \omega_U^z(O^c) \\ &\le \varepsilon + 2\|f\|_\infty\, \omega_U^z(O^c). \end{aligned}$$

Its limsup is at most ε, and then at most zero. For continuous f, the defining representation of E_U already gives equality with T_U throughout the interior. □

For a Borel set A, the indicator is continuous at each point of the relative interior of A and each point outside its relative closure. Therefore

$$\omega_U^z(A) \to 1 \quad (z \to a \in \operatorname{int}_{\partial U} A), \qquad \omega_U^z(A) \to 0 \quad (z \to a \notin \overline{A}^{\partial U}).$$

No assertion is made at the relative boundary of A. In the disk, a symmetrically approached endpoint of an arc can have limit $1/2$, while other approach geometries require the corresponding Poisson-kernel analysis. Replacing the discontinuous datum at that endpoint by either zero or one does not force the interior harmonic extension to take that value.

This extension is not an isometry on all bounded Borel functions with the pointwise supremum norm. A nonzero function supported on a common harmonic-measure null set has zero extension. On the disk, for example,

$$f = 1_{\{1\}} \ne 0, \qquad \|f\|_\infty = 1, \qquad T_{\mathbb{D}} f \equiv 0,$$

because every interior harmonic measure is absolutely continuous with respect to angular measure. Continuous data have no such ambiguity: a continuous nonzero function is nonzero on a nonempty open set, which has positive harmonic measure. That distinction explains why the continuous Dirichlet operator was an isometry while the bounded Borel extension is only a contraction.

Finally, the bounded Borel construction is a mathematical definition by integration against the already established harmonic measures. It does not assert that a discrete boundary mesh exactly represents every Borel set, or that an unbounded boundary function has an integrable extension. For an unbounded datum, integrability, convergence of truncations and finiteness must be separately established. The same caution applies to interchange of limits with measures whose base points approach the boundary: the measures concentrate, so a bound uniform in the base point is needed for conclusions stronger than the continuity-point argument proved above.

Theorem 15.12 (One-point integrability and locally uniform convergence)**.** *Fix $z_0 \in U$ and put $\mu = \omega_U^{z_0}$. If a real Borel function f satisfies $\int_{\partial U} |f| \, \mathrm{d}\mu < \infty$, then*

$$T_U f(z) = \int_{\partial U} f \, \mathrm{d}\omega_U^z$$

is finite and harmonic on U. For every compact $K \Subset U$,

$$\sup_{z\in K} |T_U f(z)| \le C_K \int_{\partial U} |f|\,\mathrm{d}\mu, \tag{15.3}$$

where C_K is the measure-comparison constant of Proposition 15.3. Let each f_n be a real Borel function on ∂U. If $f_n \to f$ μ-almost everywhere and $|f_n| \le F$ with $F \in L^1(\mu)$, then $T_U f_n \to T_U f$ locally uniformly on U. No continuous boundary trace for these unbounded data is included in the conclusion.

Proof. The Borel-set comparison $\omega_U^z(A) \le C_K \mu(A)$ first gives

$$\int s\,\mathrm{d}\omega_U^z \le C_K \int s\,\mathrm{d}\mu$$

for nonnegative simple functions s by finite addition. Increasing simple approximation and monotone convergence extend it to every nonnegative Borel function. Apply it to $|f|$ to obtain finiteness and (15.3). Sets of μ-measure zero have zero measure for every interior harmonic measure, so changing f on such a set changes none of these integrals.

Let $f^{(N)} = \max(-N, \min(f, N))$. The bounded Borel extension theorem makes $T_U f^{(N)}$ harmonic. Moreover,

$$\begin{aligned}\sup_{z\in K} |T_U f^{(N)}(z) - T_U f(z)| &\le C_K \int |f^{(N)} - f|\,\mathrm{d}\mu \\ &\le C_K \int_{\{|f|>N\}} |f|\,\mathrm{d}\mu \longrightarrow 0.\end{aligned}$$

The last limit follows from integrability, so the limit is locally uniform and preserves harmonicity. For the sequence in the statement, dominated convergence gives $\int |f_n - f|\,\mathrm{d}\mu \to 0$. Applying the same comparison to this difference gives

$$\sup_{z\in K} |T_U f_n(z) - T_U f(z)| \le C_K \int |f_n - f|\,\mathrm{d}\mu \longrightarrow 0.$$

This estimate justifies convergence with a varying interior base point inside a fixed compact set. It supplies no bound with that same constant when the base points approach the boundary. □

For a nonnegative Borel datum the resulting alternative is particularly clean. Either its integral is finite at one interior point and defines a finite nonnegative harmonic function throughout U, or its integral is $+\infty$ at every interior point. Indeed the two-sided Harnack comparison extends to nonnegative integrals by the same simple-function argument. This is an integrability statement about an already defined kernel, not an existence theorem for unrestricted signed integrals. An expression $\infty - \infty$ remains undefined. For data covered by the theorem, (15.1) also holds as an absolutely convergent identity, since its version for $|f|$ has finite value.

Investigation 15.13 (An integrable logarithmic boundary singularity)**.** On the unit circle let $f(e^{it}) = -\log|1 - e^{it}|$ for $t \neq 0$ modulo 2π, and assign any finite value at 1. Determine $T_{\mathbb{D}} f$.

Solution. For $\rho < 1$ put $f_\rho(e^{it}) = -\log|1 - \rho e^{it}|$. These continuous data have the harmonic extension

$$T_{\mathbb{D}} f_\rho(z) = -\log|1 - \rho z| = \operatorname{Re} \sum_{n\ge 1} \frac{\rho^n z^n}{n}.$$

The power series converges on a neighborhood of the closed disk for fixed $\rho < 1$, verifying the trace and harmonicity. For $1/2 \le \rho < 1$,

$$|1 - \rho e^{it}|^2 = (1-\rho)^2 + 4\rho \sin^2(t/2) \ge 2\sin^2(t/2), \qquad |1 - \rho e^{it}| \le 2.$$

Thus the negative parts of f_ρ are bounded by $\log 2$, and the positive parts are bounded by the positive part of $-\log(\sqrt{2}|\sin(t/2)|)$. This majorant is integrable: near zero $|\sin(t/2)|$ is bounded below by a positive constant times $|t|$, and $\int_0^\varepsilon -\log t\,\mathrm{d}t = \varepsilon(1-\log\varepsilon)$. Dominated convergence gives $f_\rho \to f$ in normalized angular L^1. The preceding theorem and the explicit interior limit therefore give

$$T_{\mathbb{D}} f(z) = -\log|1-z|, \qquad T_{\mathbb{D}} f(0) = 0.$$

The value assigned to the singleton 1 is irrelevant because it has zero harmonic measure. The extension diverges as $z \to 1$, which is consistent with the datum's failure of continuity there.

Investigation 15.14 (A positive datum with no finite extension)**.** Replace the logarithmic datum by $f(e^{it}) = |1-e^{it}|^{-1}$ away from 1, again assigning a finite value at that point. Show that its kernel integral is infinite at every point of the disk.

Solution. Since $|1-e^{it}| = 2|\sin(t/2)| \le |t|$ for $|t| \le \pi$, the normalized angular integral dominates a positive constant times $\int_0^\varepsilon \mathrm{d}t/t = +\infty$. For a fixed $z \in \mathbb{D}$,

$$P(z, e^{it}) \ge \frac{1-|z|}{1+|z|} > 0$$

uniformly in t. Multiplying this lower bound by the nonnegative datum proves that its Poisson integral is infinite. Changing one point of the datum cannot repair a divergence over every neighboring arc. This gives a direct disk verification of the nonnegative alternative and explains why finiteness at one point is a substantive hypothesis.

15.7 Five worked investigations

Investigation 15.15 (An exact arc probability)**.** For $A = \{e^{it} : |t| < \beta\}$ with $0 < \beta < \pi$ and $0 \le r < 1$, prove

$$\omega^r_{\mathbb{D}}(A) = \frac{2}{\pi}\arctan\!\left(\frac{1+r}{1-r}\tan\frac{\beta}{2}\right).$$

Evaluate it for $\beta = \pi/3$ at $r = 0$ and $r = 1/2$.

Solution. Start from the normalized Poisson density and use $t = 2\arctan s$. Then

$$\cos t = \frac{1-s^2}{1+s^2}, \quad \mathrm{d}t = \frac{2\,\mathrm{d}s}{1+s^2}, \quad \frac{1-r^2}{1-2r\cos t + r^2}\,\mathrm{d}t = \frac{2(1-r^2)\,\mathrm{d}s}{(1-r)^2+(1+r)^2 s^2}.$$

Integrating from $s = -\tan(\beta/2)$ to $s = \tan(\beta/2)$ and dividing by 2π gives the claimed arctangent. At the center the mass is $\beta/\pi = 1/3$. At $r = 1/2$, the arctangent argument is $3/\sqrt{3} = \sqrt{3}$, so the mass is $2/3$. The arc has unchanged angular length, but its weight doubles at this observation point.

Investigation 15.16 (Normalization as a unit test in mathematics)**.** Suppose a proposed harmonic-measure formula is $\mathrm{d}\nu_z = P(z, e^{it})\,\mathrm{d}t$. Identify the exact failure and repair it.

Solution. Set the boundary datum equal to one. The Poisson identity gives $\int_0^{2\pi} P(z, e^{it})\,\mathrm{d}t = 2\pi$, so this formula would predict the harmonic extension of one to be 2π. It fails the defining condition $E_U 1 = 1$. Replacing $\mathrm{d}t$ by $\mathrm{d}t/(2\pi)$ repairs it. Equivalently, one may redefine the kernel as $P/(2\pi)$ while keeping ordinary angular measure. Mixing these conventions would repeat the error. This symbolic normalization check is a complete calculation for the constant datum, not an empirical test of the whole representation theorem.

Investigation 15.17 (An annular boundary component)**.** Let A be the inner circle of A_ρ. Identify $\omega^z_{A_\rho}(A)$ and compute its value at $|z| = \sqrt{\rho}$.

Solution. The inner circle is clopen in the annular boundary. Its indicator therefore has a continuous harmonic extension with values one on that circle and zero on the outer one. The exact annular formula gives

$$\omega^z_{A_\rho}(A) = \frac{\log|z|}{\log\rho}, \qquad \omega^z_{A_\rho}(A) = \frac{1}{2} \quad \text{when } |z| = \sqrt{\rho}.$$

The complementary outer-circle mass is one minus this quantity. Both are strictly between zero and one in the annulus. This example shows why harmonic measure may denote either a measure at fixed z or, after a boundary set is fixed, a harmonic function of z; the fixed variable should always be stated.

Investigation 15.18 (A small exceptional arc can dominate an error)**.** Suppose boundary data on the disk differ by at most M on $A = \{|t| < \pi/3\}$ and agree outside A. Compare the resulting pointwise error bounds at 0 and $1/2$.

Solution. Proposition 15.7 with $\delta = 0$ gives

$$|E_{\mathbb{D}}g(0) - E_{\mathbb{D}}k(0)| \le M/3, \qquad |E_{\mathbb{D}}g(1/2) - E_{\mathbb{D}}k(1/2)| \le 2M/3.$$

For continuous data the abrupt indicator of the open arc is not itself an allowed continuous perturbation, but continuous functions supported inside that arc can approximate its mass monotonically. Hence these bounds have the expected sharp limiting interpretation. The global uniform bound remains M; the refined interior bounds encode the location of the error rather than only its maximum size.

Investigation 15.19 (Weak convergence does not preserve individual point masses)**.** On the unit disk each harmonic measure has zero mass at every boundary singleton. Explain how $\omega^r_{\mathbb{D}} \to \delta_1$ as $r \uparrow 1$ is consistent with that fact.

Solution. For every fixed $r < 1$, the Poisson density is continuous with respect to angular measure, so $\omega^r_{\mathbb{D}}(\{1\}) = 0$. Weak convergence tests continuous boundary functions, not the discontinuous indicator of a singleton. Thus

$$\int g \, \mathrm{d}\omega^r_{\mathbb{D}} \longrightarrow g(1) \quad (g \in C(\partial\mathbb{D}))$$

does not imply convergence of the masses of $\{1\}$. On every fixed open arc around 1, the explicit arc formula gives mass tending to one, while the density grows and concentrates in a shrinking neighborhood. The limiting measure can therefore have an atom even though every measure in the approximating family is atomless.

Chapter 16

Green functions with a controlled logarithmic pole

Source correspondence. Lyubich, Section 10.9, Remarks 10.21–10.22, and Theorem 10.23.

A Green function combines three pieces of information that should not be separated: the strength of its singularity, the boundary condition, and the sign of the Laplacian. Throughout these chapters the convention is

$$G_U(z,p) > 0, \qquad G_U(z,p) = -\log|z-p| + O(1), \qquad -\Delta_z G_U(z,p) = 2\pi\delta_p. \tag{G}$$

The last equality is distributional, and the coordinate expression for its singularity is local. A coordinate change adds a bounded harmonic term to the logarithm. On a Riemann surface the invariant formulation is $-dd^c G_U = 2\pi\delta_p$, where the right side is a measure, or a distributional two-form. A compatible metric may be chosen for gradients and normal derivatives; the assertions concerning harmonicity and energy do not depend on that choice.

16.1 The exact existence contract

Definition 16.1 (Regular Green domain)**.** Let S be a connected Riemann surface. A regular Green domain in this chapter is a connected open set $U \subset S$ with compact closure, nonempty boundary, and Dirichlet-regular boundary at every point. A coefficient-one Green function with pole $p \in U$ is a function $G : U \setminus \{p\} \to \mathbb{R}$ that is harmonic there, tends to zero at every point of ∂U, and satisfies, in a coordinate $w(p) = 0$,

$$G(w) + \log|w| = O(1) \quad (w \to 0).$$

The value at the pole is interpreted as $+\infty$, not as a finite harmonic value.

The nonempty-boundary condition is substantive. A compact surface without boundary cannot support a function with just this positive pole and distributional Laplacian $-2\pi\delta_p$: testing against the constant function would give $0 = -2\pi$. Green kernels on closed surfaces instead require a compensating constant density and a normalization for additive constants. Those are different boundary-value problems.

Lemma 16.2 (Bounded remainder and uniqueness)**.** *If G satisfies Definition 16.1, then*

$$G(w) = -\log|w| + h(w)$$

near its pole, where h is harmonic on a full coordinate disk. Such a Green function is unique and is strictly positive off its pole.

Proof. The difference $h = G + \log|w|$ is harmonic on the punctured disk and bounded there. The bounded harmonic removal theorem extends it harmonically across 0. In particular its first derivatives remain bounded on a smaller disk, so

$$G(w) = -\log|w| + h(0) + O(|w|), \qquad \partial_r G = -r^{-1} + O(1). \tag{1}$$

If G_1, G_2 are two candidates, their difference has a removable singularity and zero boundary limits. For $\epsilon > 0$, compactness of ∂U gives a boundary neighborhood where $|G_1 - G_2| < \epsilon$. The remaining compact set admits a maximum and a minimum. A positive maximum exceeding ϵ would therefore be an interior maximum of a harmonic function. The maximum principle excludes it unless the difference is constant; its boundary values then force that constant to be zero. Applying this to both signs and letting ϵ decrease to zero proves uniqueness.

Here compactness supplies a uniform boundary neighborhood even though the definition gives separate limits at separate points. For each boundary point ξ, choose an open neighborhood N_ξ on which the absolute difference is below ϵ. A finite subcover of ∂U has union N with that same bound, and $\overline{U} \setminus N$ is compact and contained in U. Thus any value exceeding ϵ in absolute value produces an attained interior extremum on a compact set. This is the exact passage from pointwise boundary limits to the global maximum argument; no smooth boundary or boundary parametrization is used.

For positivity, choose a sufficiently small coordinate circle on which $G > 0$, using (1). The minimum principle on the domain outside that circle, with outer boundary limit zero, gives $G \geq 0$. On the punctured coordinate disk the same conclusion follows by comparison between the fixed circle and smaller circles where G is positive. The punctured domain is connected, and a zero in it would make G constant by the strong minimum principle. Its pole rules out constancy. Hence $G > 0$ everywhere away from p. □

16.2 A planar construction and why it is not a surface proof

Suppose first that U is a bounded regular domain in $\mathbb{C}$. The boundary function $\zeta \mapsto -\log|\zeta - p|$ is continuous, because $\operatorname{dist}(p, \partial U) > 0$. Let H be its harmonic extension. Then

$$G_U(z, p) = -\log|z - p| - H(z). \tag{2}$$

Every condition in the definition follows directly: both terms are harmonic away from p, the boundary values cancel, and H is bounded near p. Formula (2) also shows precisely what is required of the Dirichlet solver. It need not solve a singular boundary problem; the pole has already been isolated in a known elementary function.

On a general surface there is no global coordinate $z - p$. A coordinate cutoff of $-\log|w|$ creates a smooth forcing term on its transition annulus; solving for that forcing requires a Poisson theory that is not yet available. The following construction avoids assuming the result that it is meant to establish.

Theorem 16.3 (Green existence on a regular surface domain)**.** *Every regular Green domain $U \Subset S$ has a unique coefficient-one Green function at each $p \in U$.*

Proof. Choose a coordinate disk $|w| < R_0$ whose closure lies in U, and choose $0 < r < R < R_0$. Extend

$$u_0(w) = \max\{\log(r/|w|), 0\}$$

by zero outside $|w| < r$. This is continuous and subharmonic on $U \setminus \{p\}$: near the joining circle it is the maximum of the harmonic functions $\log(r/|w|)$ and 0. Its pole coefficient is one.

Let $\mathcal{P}$ consist of continuous subharmonic functions u on $U \setminus \{p\}$ such that

$$\limsup_{z\to\xi} u(z) \le 0 \quad (\xi \in \partial U), \qquad u(w) = -\log|w| + O_u(1), \qquad u \ge u_0. \tag{3}$$

The subscript on $O_u(1)$ matters: at this stage its constant can depend on the candidate. The family is nonempty, closed under finite maxima, and closed under harmonic replacement on coordinate disks avoiding the pole. A replacement increases u, preserves its boundary restrictions, and changes nothing near the pole if the disk is chosen away from it. Thus it is a Perron family in the sense of Theorem 11.3.

For clarity, the closure statements retain the prescribed coefficient, not merely the existence of some logarithmic singularity. If $u_i = -\log|w| + b_i(w)$ with bounded b_i near zero, then

$$\max(u_1, u_2) = -\log|w| + \max(b_1, b_2),$$

and the new remainder is still bounded. Its boundary upper limits stay nonpositive, and its lower bound by u_0 is unchanged. For harmonic replacement on a disk $D \Subset U \setminus \{p\}$, solve the continuous Dirichlet problem with datum $u|_{\partial D}$. Comparison gives a harmonic replacement at least as large as u on D. Pasting across ∂D preserves subharmonicity, while the function is unchanged on a neighborhood of the pole and on a neighborhood of ∂U. Therefore every condition in (3) survives the replacement. These checks are what allow the Perron theorem to be applied to the singularity- constrained family rather than to a larger family with uncontrolled pole strength.

We now obtain a bound independent of u. Write

$$M_s(u) = \max_{|w|=s} u(w), \qquad s = r, R.$$

These are signed maxima, not maxima of absolute values. They are nonnegative because $u \ge u_0 \ge 0$. On $0 < |w| < R$ the function $u + (1+\epsilon)\log|w|$ is subharmonic and tends to $-\infty$ at the pole. Comparison on annuli, followed by shrinking their inner boundary, therefore gives

$$u(w) + (1+\epsilon)\log|w| \le M_R(u) + (1+\epsilon)\log R.$$

Letting $\epsilon \downarrow 0$ yields both

$$M_r(u) \le M_R(u) + \log(R/r), \qquad u(w) \le M_R(u) + \log(R/|w|). \tag{4}$$

For the complementary comparison, remove the closed coordinate disk $|w| \le r$. By Lemma 13.4, its complement in U is connected and has regular boundary. Let v solve the Dirichlet problem there with values one on $|w| = r$ and zero on ∂U. The maximum principle gives $0 < v < 1$. In particular

$$\lambda = \max_{|w|=R} v(w) < 1. \tag{5}$$

The function $u - M_r(u)v$ is subharmonic on this complementary domain. Its boundary upper limits are nonpositive on both boundary pieces. Comparison therefore gives

$$u \le M_r(u)v, \qquad M_R(u) \le \lambda M_r(u). \tag{6}$$

Combining (4) and (6), without dividing by a possibly negative quantity, gives the uniform bound

$$0 \le M_r(u) \le C := \frac{\log(R/r)}{1-\lambda}. \tag{7}$$

Consequently $u \le Cv$ outside the small disk; inside it, (4) gives $u(w) \le \lambda C + \log(R/|w|)$. Every compact subset of the punctured domain is therefore uniformly bounded above over the family.

The Perron theorem now implies that $G = \sup_{u \in \mathcal{P}} u$ is finite and harmonic away from p. The two comparisons give

$$0 \le G(z) \le Cv(z) \quad (z \in U \setminus \overline{D(p,r)}),$$

so G tends to zero at every boundary point. Near the pole,

$$\log(r/|w|) \le G(w) \le \lambda C + \log(R/|w|). \tag{8}$$

Thus $G + \log|w|$ is bounded. The preceding lemma supplies its harmonic extension, strict positivity, and uniqueness. All constants used to prove finiteness have been chosen before taking the supremum; that is the essential compactness point in the construction. □

16.2.1 The constants left after taking the envelope

The existence proof yields quantitative local information in addition to finiteness. Keep its coordinate radii $0 < r < R < R_0$, comparison function v, number $0 < \lambda < 1$, and constant C. For every candidate and for its envelope the same estimates hold:

$$\begin{aligned} M_r &\le M_R + \log(R/r), & M_R &\le \lambda M_r, \\ (1-\lambda)M_r &\le \log(R/r), & 0 \le G &\le Cv \quad \text{outside } \overline{D}_r. \end{aligned}$$

The strict inequality $\lambda < 1$ follows on a compact circle entirely inside the connected punctured domain. If equality held at any point of that circle, the strong maximum principle would make v identically one, contrary to its zero outer boundary values. This explains why the denominator is positive without assuming a numerical lower bound for it. When λ approaches one, this particular estimate deteriorates; the deterioration is a property of the comparison construction, not a failure of existence.

Write $h(w) = G(w) + \log|w|$ on D_r. The removable extension and (8) give, including the center by continuity,

$$\log r \le h(w) \le \lambda C + \log R, \qquad |h(w)| \le H := \max\{|\log r|, |\lambda C + \log R|\}.$$

For $|w| \le r/2$, apply the center derivative formula from the Poisson integral on the radius-s disk about w, with $s = r/2$. In any unit direction e, rotation of the angular variable gives

$$|\partial_e h(w)| \le \frac{1}{\pi s} \int_0^{2\pi} H|\cos\theta|\, d\theta = \frac{4H}{\pi s} = \frac{8H}{\pi r} =: A_1.$$

Taking the supremum over directions proves $|\nabla h| \le A_1$. Integration along the segment from zero to w then supplies the explicit remainder estimates

$$|h(w) - h(0)| \le A_1|w|, \qquad \left|\partial_s G(se^{i\theta}) + \frac{1}{s}\right| \le A_1 \quad (0 < s \le r/2).$$

These inequalities justify the constants hidden in the earlier pole expansion uniformly in angle. They also identify the information needed to turn an excision limit into a finite-radius error estimate.

There is a useful stability distinction. In the planar construction (2), if an approximate harmonic correction $\widehat{H}$ has boundary error at most η, then the maximum principle gives

$$|(-\log|z-p| - \widehat{H}(z)) - G_U(z,p)| \le \eta \quad (z \ne p).$$

The singularities cancel before the estimate is made. This bound requires the correction to be harmonic; an arbitrary sampled correction also needs a residual estimate. Likewise, the surface construction gives actual analytic constants C, H, A_1 once its comparison solution is known, but it does not by itself constitute an executed numerical evaluation of those constants.

16.3 The point mass and the zero extension

Proposition 16.4 (Distributional sign)**.** *In a coordinate disk, $\Delta \log|w| = 2\pi\delta_0$. Therefore the Green function of Theorem 16.3 satisfies $\Delta G = -2\pi\delta_p$ in the interior.*

Proof. Let ψ be smooth and compactly supported in the disk. On its complement of $|w| \le \epsilon$, Green's identity gives

$$\int_{|w|>\epsilon} \log|w|\,\Delta\psi\,dA = \int_{|w|=\epsilon} \left(-\log\epsilon\,\partial_r\psi + \frac{\psi}{\epsilon}\right) ds.$$

The outward normal of the punctured region points toward the deleted disk, which accounts for both signs. The first boundary integral is $O(\epsilon|\log\epsilon|)$ and tends to zero. The second is $\int_0^{2\pi} \psi(\epsilon e^{i\theta})\,d\theta$, which tends to $2\pi\psi(0)$. Local integrability of the logarithm justifies the limit on the left. The harmonic remainder in (1) has zero distributional Laplacian, proving the assertion. □

Extending G by zero outside U produces a continuous subharmonic function locally away from the pole: on a neighborhood not containing p, the positive harmonic function inside is pasted to zero with the subharmonic orientation. At the pole the extension takes $+\infty$ and is not subharmonic under the usual definition. For a smooth planar boundary its full distributional identity is

$$\Delta\widetilde{G} = -2\pi\delta_p - (\partial_n G)\,ds\big|_{\partial U}, \tag{9}$$

where n points out of U and $\partial_n G \le 0$. The boundary term is positive, while the interior pole term is negative. To verify (9), apply Green's identity to $\int_U G\Delta\psi$, use $G = 0$ on the boundary, and keep the pole contribution from the preceding proposition. The boundary measure has total mass 2π, so the two terms balance when tested against a function equal to one near $\overline{U}$.

The source's suggestion that $-G$ is automatically a barrier at every individual boundary point also needs care. It vanishes at all boundary points, and hence does not by itself have the point-separation property of the strict barriers used in the Dirichlet construction. We use regularity as an explicit hypothesis here; no converse is inferred from that incomplete barrier argument.

16.4 Solved investigations

Investigation 16.5 (Changing the pole strength)**.** Replace the singularity in Definition 16.1 by $-a\log|w| + O(1)$, where $a > 0$. Determine the solution and its Laplacian.

Solution. The solution is aG_U. Its difference from any other candidate is bounded harmonic at the pole and zero on the boundary, so uniqueness is unchanged. Linearity gives

$$-\Delta(aG_U) = 2\pi a\delta_p.$$

If $a = 0$, the unique solution is zero and strict positivity disappears. For $a < 0$ the unique solution is negative. Thus the word "positive" encodes the sign of the source, not a separate choice after construction.

Investigation 16.6 (A coordinate change at the pole)**.** Suppose $\eta = cw + O(w^2)$, $c \neq 0$. If $G = -\log|w| + h_w(0) + o(1)$, find the regular part in the η coordinate.

Solution. Since $|\eta|/|w| \to |c|$,

$$G + \log|\eta| = h_w(0) + \log|c| + o(1).$$

Therefore $h_\eta(0) = h_w(0) + \log|c|$. The coefficient of the logarithm is invariant, whereas the numerical regular part depends on scale. This is why logarithms of physical lengths require a reference length.

Investigation 16.7 (Domain monotonicity)**.** Let $V \subset U$ be regular Green domains and $p \in V$. Show that $G_V(z,p) \le G_U(z,p)$ for $z \in V \setminus \{p\}$.

Solution. The difference $G_V - G_U$ has a removable singularity at p. At points of ∂V lying in U its upper limit is $-G_U \le 0$; at points on ∂U it is zero. The maximum principle gives the claim. In the concentric disks $D_r \subset D_R$ with pole zero, the difference is exactly

$$G_{D_R}(z,0) - G_{D_r}(z,0) = \log(R/r) > 0.$$

The strict inequality expresses the extra potential sustained when the grounded boundary is moved farther from the source.

Investigation 16.8 (A bounded puncture is not a grounded boundary)**.** Explain why the punctured unit disk does not have a Green function that vanishes at every boundary point, with pole $p \neq 0$.

Solution. If it did, the zero limit at 0 would make the singularity there removable. The extended function would be nonnegative harmonic near 0 and equal zero at 0. The strong minimum principle would force it to be zero on the connected disk minus p, contradicting its positive pole. The disk Green function still exists on the punctured disk as a potential-theoretic kernel, but its positive limit at 0 does not solve the all-boundary-point Dirichlet problem. Regularity is precisely the missing hypothesis.

Investigation 16.9 (Flux through a small circle)**.** Calculate $-\int_{|w|=\epsilon} \partial_r G \, ds$ and explain its limit.

Solution. From (1), the integral is $2\pi + O(\epsilon)$, and in fact it is exactly 2π once the circle lies in the harmonic-remainder chart:

$$-\int_{|w|=\epsilon} \partial_r G \, ds = 2\pi - \int_{|w|=\epsilon} \partial_r h \, ds = 2\pi - \int_{|w|<\epsilon} \Delta h \, dA = 2\pi.$$

The logarithmic source strength can therefore be read from any sufficiently small enclosing circle, independently of its radius.

Chapter 17

Reciprocity, boundary flux, and conformal transport

17.1 Green's identities with the normals visible

For a bounded planar domain V with piecewise smooth boundary and $u, v \in C^2(\overline{V})$, the divergence theorem applied to $u\nabla v$ and to $u\nabla v - v\nabla u$ gives

$$\int_V \nabla u \cdot \nabla v \, dA = \int_{\partial V} u \partial_n v \, ds - \int_V u \Delta v \, dA,$$
$$\int_V (u\Delta v - v\Delta u) \, dA = \int_{\partial V} (u\partial_n v - v\partial_n u) \, ds. \tag{10}$$

There is only one convention: n is the outward normal of the region over which the integral is taken. On a hole it points into the hole. For a conformal metric $ds_g^2 = \rho^2(dx^2 + dy^2)$,

$$\Delta_g = \rho^{-2}\Delta, \quad dA_g = \rho^2 dA, \quad \partial_{n_g} = \rho^{-1}\partial_n, \quad ds_g = \rho ds. \tag{11}$$

Thus the combinations in (10) are invariant. A partition into coordinate patches gives the same identities on a surface, with internal patch boundary integrals canceling in pairs.

When a domain has analytic boundary arcs and a harmonic function has zero continuous boundary values, its local odd reflection across each arc is harmonic. One way to verify this without presupposing derivative regularity is to flatten the arc conformally, solve the Dirichlet problem on a small full disk using odd boundary data, and compare on the half disk. Uniqueness identifies the original function with the restriction of this smooth harmonic extension. Consequently the Green kernel is smooth up to such boundary arcs away from its pole. In the next proposition this is the regularity needed to use (10) literally.

Proposition 17.1 (Reciprocity on a smooth bordered domain)**.** *If V is a relatively compact surface domain bounded by finitely many analytic curves, then*

$$G_V(p, q) = G_V(q, p) \qquad (p, q \in V,\ p \neq q).$$

Proof. Set $u(z) = G_V(z, p)$ and $v(z) = G_V(z, q)$. Remove disjoint coordinate disks of radius ϵ about the two poles. Both functions are harmonic on the remaining region; their outer boundary values are zero. Green's second identity therefore reduces to integrals over the two small circles.

Near p the expansions are

$$u = -\log r + O(1), \quad \partial_n u = r^{-1} + O(1), \quad v = v(p) + O(r), \quad \partial_n v = O(1),$$

where $n = -\partial_r$ on the deleted-circle boundary. Hence

$$\int_{|w|=\epsilon} (u\partial_n v - v\partial_n u)\, ds = -2\pi v(p) + O(\epsilon|\log\epsilon|) + O(\epsilon).$$

Near q the roles reverse and the limit is $2\pi u(q)$. The sum is zero by (10), so $u(q) = v(p)$. The calculation also identifies why a naive substitution into Green's identity on the unpunctured domain is invalid: each pole contributes a finite nonzero boundary term. □

17.1.1 A finite-radius version of the pole calculation

The preceding limit can be organized into three separately bounded terms. In a coordinate disk about p, let $u = -\log r + h$ and let v be harmonic without a pole there. Suppose on the closed radius-r_* disk that

$$|h| \le H, \quad |\nabla h| \le A_1, \quad |v| \le B_0, \quad |\nabla v| \le B_1, \qquad 0 < r_* < 1.$$

On a removed circle of radius $0 < r \le r_*$, put $I_r = \int (u\partial_n v - v\partial_n u)\, ds$, with $n = -\partial_r$. Because $\partial_n u = r^{-1} - \partial_r h$ and $ds = r\, d\theta$,

$$\begin{aligned} I_r + 2\pi v(p) &= \int_{|w|=r} u\partial_n v\, ds - \int_0^{2\pi} (v(re^{i\theta}) - v(p))\, d\theta + \int_{|w|=r} v\partial_r h\, ds, \\ |I_r + 2\pi v(p)| &\le 2\pi r\{B_1(-\log r + H) + B_1 + B_0 A_1\}. \end{aligned}$$

The first term is bounded using $|u| \le -\log r + H$; the second uses $|v(w) - v(p)| \le B_1 r$; the third uses the bounded regular derivative. Thus the error tends to zero with an explicit $r|\log r|$ majorant. No cancellation of two divergent integrals has been assumed.

For two poles choose disjoint fixed coordinate disks and perform this calculation in each. The sum of the two displayed error bounds controls the finite-radius reciprocity defect. In a conformal metric the product of normal derivative and arclength is unchanged by (11), so these same coordinate constants suffice; an extra metric-density factor must not be inserted into the boundary integral.

One can make every constant elementary in the unit disk. Set $p = 0.35$, $q = -0.3 + 0.2i$, $r_* = 0.10$, and $d = |p - q| - r_* > 0$. For $u = G_{\mathbb{D}}(\cdot, p)$ and $v = G_{\mathbb{D}}(\cdot, q)$, valid choices are

$$H = -\log(1 - p(p + r_*)), \qquad A_1 = \frac{p}{1 - p(p + r_*)},$$

$$B_1 = \frac{|q|}{1 - |q|(p + r_*)} + \frac{1}{d}, \qquad B_0 = |G_{\mathbb{D}}(p, q)| + r_* B_1.$$

Indeed, $h(z) = \log|1 - pz|$ and $|\nabla h(z)| = p/|1 - pz|$. The two terms of v have gradient norms at most $|q|/|1 - \overline{q}z|$ and $1/|z - q|$. The triangle inequality and $|z| \le p + r_*$ give the stated constants. This explicit example is a finite-radius estimate proved from exact formulas; the plotted curves below do not supply an experimental substitute for the bounds.

For exact harmonic data on a complete coordinate circle there is a stronger cancellation. Write $u = -\log r + h$ as above. Harmonicity on the full disk gives

$$\int_{|w|=r} \partial_n v\, ds = 0, \qquad \int_0^{2\pi} v(re^{i\theta})\, d\theta = 2\pi v(p), \qquad \int_{|w|=r} (h\partial_n v - v\partial_n h)\, ds = 0.$$

The first equality is flux conservation, the second is the mean-value identity, and the third is Green's second identity for the nonsingular pair h, v, with the reversed normal changing both terms together. Since $-\log r$ is constant on the circle, substitution yields $I_r = -2\pi v(p)$ exactly at every admissible radius. The positive majorant above is therefore the conservative termwise

estimate obtained before these exact cancellations. It displays the scales of separate boundary contributions; it is not a claim of a nonzero defect in the ideal analytic identity. Approximate boundary factors require their own error bounds before any such cancellation can be used in a computation.

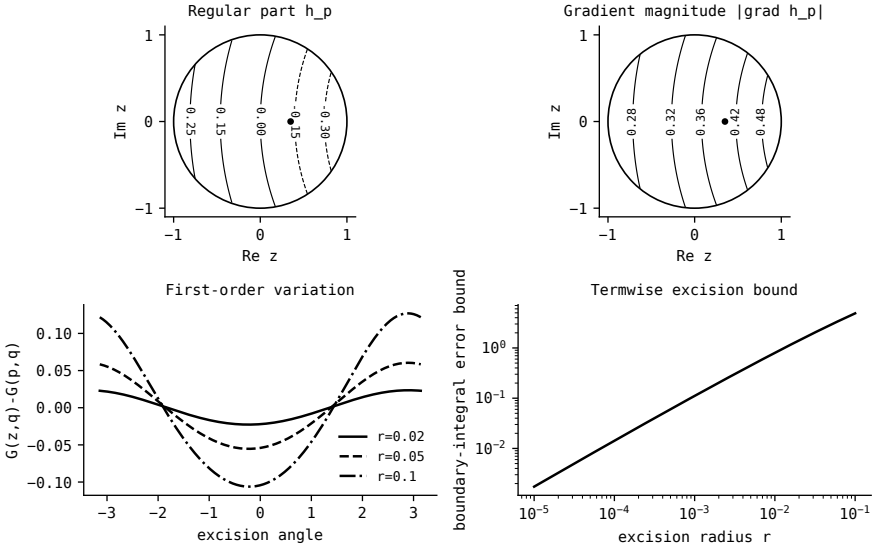


Figure 17.1: Pole regularization and finite-circle control in the unit disk. The first panels show $h_p(z) = \log|1 - 0.35z|$ and $|\nabla h_p| = 0.35/|1 - 0.35z|$. Angular profiles compare $G_{\mathbb{D}}(p + re^{i\theta}, q)$ with $G_{\mathbb{D}}(p, q)$ for $p = 0.35$, $q = -0.3 + 0.2i$, and $r = 0.02, 0.05, 0.10$. The final panel shows the proved excision-error majorant from this subsection for $10^{-5} \le r \le 0.10$, using its explicit constants. Sampled profiles illustrate exact analytic functions; the majorant is an inequality for the complete circle.

17.2 Passing from smooth boundaries to regular boundaries

Lemma 17.2 (Green-level exhaustion)**.** *Fix $p \in U$ in a regular Green domain. There is a decreasing sequence $t_j \downarrow 0$ such that*

$$U_j = \{G_U(\cdot, p) > t_j\} \cup \{p\}$$

is connected, has compact closure in U, has finitely many analytic boundary curves, and increases to U. For any $q \in U$, once $q \in U_j$,

$$G_{U_j}(z, q) \uparrow G_U(z, q)$$

locally uniformly away from q.

Proof. The zero boundary limits and compactness of ∂U imply that a positive superlevel stays away from ∂U. The pole expansion shows that its complement does not contain points arbitrarily close to p. The critical points of G away from p are isolated: in a local chart $G_x - iG_y$ is holomorphic, and an accumulation of its zeros would make G constant on the connected punctured domain. There are only countably many critical points, because a surface has a countable base. Choose t_j to avoid their values.

Each level $G = t_j$ is then a compact analytic one-dimensional manifold. It has finitely many components: a finite collection of local level charts covers it, and each chart meets only its one

local arc after shrinking. Each component is a circle. A component of $\{G > t_j\}$ not containing the pole would have compact closure and boundary value t_j; the maximum principle would contradict its strict inequality. Thus U_j is connected. Strict positivity gives $\bigcup_j U_j = U$.

Domain monotonicity bounds the increasing kernels by $G_U(\cdot, q)$. Harnack convergence supplies a harmonic limit K away from q. A compact set $L \Subset U \setminus \{q\}$ lies in U_j for all sufficiently large j: otherwise choose points $z_j \in L \setminus U_j$, pass to a convergent subsequence, and use positivity and continuity of $G_U(\cdot, p)$ at its limit, or its divergence if that limit is p, to contradict $G_U(z_j, p) \le t_j$. On a slightly larger compact neighborhood the upper bound $G_U(\cdot, q)$ is finite. Interior derivative estimates therefore give equicontinuity of the increasing kernels on L. Any uniformly convergent subsequence has their pointwise supremum as its limit, so all subsequences have the same limit; this supplies a direct compactness explanation for the locally uniform convergence used here. A fixed initial kernel is a lower bound near q, so

$$G_{U_{j_0}}(z, q) \le K(z) \le G_U(z, q)$$

shows that $K + \log|w|$ is bounded at the pole. At the outer boundary, $0 \le K \le G_U(\cdot, q)$ forces the zero limit. Uniqueness therefore gives $K = G_U(\cdot, q)$. Local uniform convergence follows from the same Harnack convergence theorem, or from monotonicity and compactness. □

Theorem 17.3 (General reciprocity)**.** *On every regular Green domain, $G_U(p, q) = G_U(q, p)$ for distinct interior points.*

Proof. Choose a Green-level exhaustion containing both points from some index onward. Each exhausted domain has the boundary regularity needed for Proposition 17.1. Therefore

$$G_U(p, q) = \lim_j G_{U_j}(p, q) = \lim_j G_{U_j}(q, p) = G_U(q, p).$$

No normal derivative on the original, possibly irregularly shaped boundary has been introduced. The limit argument is what permits the general regular boundary in this theorem. □

17.3 Representation and harmonic-measure density

Theorem 17.4 (Green representation)**.** *Let V be a bounded planar domain with analytic boundary, and let $u \in C^2(\overline{V})$ solve $-\Delta u = f$. For $p \in V$,*

$$u(p) = \frac{1}{2\pi} \int_V G_V(z, p) f(z)\, dA(z) - \frac{1}{2\pi} \int_{\partial V} u \partial_n G_V(\cdot, p)\, ds. \tag{12}$$

In particular harmonic measure has the density

$$d\omega^p(\zeta) = -\frac{1}{2\pi} \partial_{n_\zeta} G_V(\zeta, p)\, ds(\zeta). \tag{13}$$

Proof. Apply (10) with $v = G_V(\cdot, p)$ after removing a coordinate disk at p. The interior integrand is $G_V f$, because $\Delta G_V = 0$ off the pole and $\Delta u = -f$. The outer boundary contributes $u \partial_n G_V$, since $G_V = 0$ there. On the inner circle,

$$\int (u \partial_n G_V - G_V \partial_n u)\, ds = 2\pi u(p) + O(\epsilon) + O(\epsilon |\log \epsilon|).$$

Thus $\int_V G_V f = \int_{\partial V} u \partial_n G_V + 2\pi u(p)$, which is (12). The logarithm is locally integrable, so the volume integral converges.

For harmonic u, (12) represents evaluation by a boundary measure. Positivity of G_V in the interior and zero boundary values imply $\partial_n G_V \le 0$. Taking $u = 1$ shows that the measure in

(13) has total mass one. For real-analytic boundary data the harmonic extension is smooth up to the boundary: locally flatten an analytic arc, extend its real-analytic datum as the real part of a holomorphic function, and subtract that function from the harmonic solution. The difference has zero boundary values and therefore has the harmonic reflection established above. Thus (12) applies to this class of harmonic extensions.

This class is uniformly dense in the continuous boundary data. Each of the finitely many analytic boundary circles has a periodic analytic parametrization; its continuous data can be uniformly approximated by trigonometric polynomials. For an explicit approximation, convolve with the normalized Fejér kernel

$$F_N(t) = \frac{1}{N}\left(\frac{\sin(Nt/2)}{\sin(t/2)}\right)^2, \qquad \frac{1}{2\pi}\int_0^{2\pi} F_N(t)dt = 1.$$

It is nonnegative and has mass tending to zero uniformly outside every fixed neighborhood of zero, since there $F_N \le C/N$. Splitting the convolution into that neighborhood and its complement proves uniform convergence from uniform continuity. The resulting trigonometric polynomials are real analytic in the boundary parameter. Apply the approximation separately to the finitely many components. The maximum principle and unit mass then pass the representation identity to every continuous boundary datum, proving (13) without an unproved general smooth-boundary regularity assertion. □

Proposition 17.5 (Conformal invariance)**.** *Let $F : U \to V$ be a conformal isomorphism between regular Green domains. Then*

$$G_U(z,p) = G_V(F(z),F(p)). \tag{14}$$

If F extends to a boundary homeomorphism, harmonic measures satisfy $F_*\omega_U^p = \omega_V^{F(p)}$.

Proof. The right side is harmonic away from p. Since $F'(p) \neq 0$, its singularity is $-\log|z-p|+O(1)$. To verify its boundary limit without assuming a boundary extension, let z_j tend to ∂U. If $F(z_j)$ stayed in a compact subset of V, a convergent subsequence and continuity of F^{-1} would force a subsequence of z_j to converge inside U, a contradiction. Thus $F(z_j)$ leaves every compact subset of V. Relative compactness and the zero boundary limits imply $G_V(F(z_j),F(p)) \to 0$. Uniqueness proves (14).

For the measure statement, compose a harmonic extension on V with F and use uniqueness on U. For every continuous g on ∂V,

$$\int_{\partial V} g\, d\omega_V^{F(p)} = \int_{\partial U} (g\circ F)\, d\omega_U^p.$$

This is exactly the equality of the pushforward measures. □

17.4 Constructing a solution with distributed forcing

Representation (12) identifies an existing smooth solution. Existence for a smooth source can also be proved, without differentiating a singular integral twice by an unjustified rule. Let U be a bounded regular planar domain and let $f \in C_c^\infty(U)$. Extend f by zero to the plane and set

$$N_f(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} -\log|y|\, f(x-y)\, dA(y). \tag{14a}$$

For x in a fixed compact set, the integrand is supported in a fixed bounded set of y. The logarithm is locally integrable there, and every derivative of f is bounded. Dominated differentiation, applied with the derivative on f, gives

$$\partial^\alpha N_f(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} -\log|y|\, \partial^\alpha f(x-y)\, dA(y).$$

Thus N_f is smooth. Proposition 16.4, tested against the translated smooth function $f(x-\cdot)$, gives

$$-\Delta N_f(x) = f(x).$$

Let H be the harmonic extension to U of the continuous boundary function $N_f|_{\partial U}$. Then $u = N_f - H$ satisfies

$$-\Delta u = f \text{ in } U, \qquad u|_{\partial U} = 0. \tag{14b}$$

It is smooth in the interior and continuous on the closure. The maximum principle proves uniqueness for this problem, because the difference of two solutions is harmonic with zero boundary values. This argument uses a global planar logarithm and is therefore stated for planar domains, unlike the surface Green-existence theorem.

On a domain with analytic boundary, the constructed solution is smooth near the boundary because f vanishes there and the zero-boundary harmonic function reflects locally. Applying (12) now proves

$$u(p) = \frac{1}{2\pi}\int_U G_U(z,p) f(z)\, dA(z). \tag{14c}$$

The kernel formula is consequently both a representation and a solution construction in this smooth compact-source class. If $f \geq 0$, positivity of the kernel gives $u \geq 0$. For signed sources the same formula gives

$$|u(p)| \leq \frac{1}{2\pi}\int_U G_U(z,p)|f(z)|\, dA(z).$$

An extension to less regular f requires the corresponding solution concept and regularity estimates. For example a bounded source naturally gives a distributional solution, but the displayed classical C^2 conclusion cannot be inferred by differentiating an arbitrary bounded function as though it were smooth. The present construction deliberately states the class in which each operation is justified.

17.5 Solved investigations

Investigation 17.6 (Where the factor 2π goes)**.** A reference defines K by $-\Delta K = \delta_p$. Translate its representation formula to the present convention.

Solution. $K = G/(2\pi)$, so

$$u(p) = \int_V K f\, dA - \int_{\partial V} u\partial_n K\, ds = \frac{1}{2\pi}\left(\int_V Gf\, dA - \int_{\partial V} u\partial_n G\, ds\right).$$

One must change both the volume kernel and the boundary flux. Changing only the logarithm would give a harmonic-measure density of mass 2π instead of one.

Investigation 17.7 (A representation check with nonzero forcing)**.** In the unit disk let $u(z) = 1 - |z|^2$. Verify (12) at $p = 0$.

Solution. Here $f = 4$, the boundary term is zero, and $G(z,0) = -\log|z|$. Using $\int_0^1 -r\log r\, dr = 1/4$ gives

$$\frac{1}{2\pi}\int_{\mathbb{D}} 4(-\log|z|)\, dA = 4\int_0^1 -r\log r\, dr = 1 = u(0).$$

The sign is positive because a positive source for $-\Delta$ with a grounded boundary produces a positive solution.

Investigation 17.8 (Boundary scaling)**.** Let $V = RU$ for $R > 0$. Relate the flux density on ∂V to that on ∂U.

Solution. From (14), $G_{RU}(Rz, Rp) = G_U(z, p)$. Differentiation gives $\partial_{n_{Rz}} G_{RU} = R^{-1}\partial_{n_z} G_U$, while $ds_{Rz} = R\, ds_z$. Thus the density per unit length scales by R^{-1}, and its measure is invariant. A charge probability is dimensionless; a density per length is not.

Investigation 17.9 (Reciprocity with two distributed sources)**.** Let $-\Delta u = f$, $-\Delta v = g$ with zero boundary data and smooth sources. Prove $\int_V fv = \int_V gu$.

Solution. Green's first identity gives both integrals as $\int_V \nabla u \cdot \nabla v$. Equivalently, the representation formula and symmetry give

$$\int_V fv = \frac{1}{2\pi} \int_V \int_V f(x) G_V(x, y) g(y)\, dA_y\, dA_x = \int_V gu.$$

The logarithmic singularity is integrable and the sources are bounded, so the double integral is absolutely convergent. This is the continuous analogue of symmetry of a conductance matrix.

Investigation 17.10 (Why a density formula needs regularity)**.** Can (13) be asserted on every Dirichlet-regular boundary merely because harmonic measure exists?

Solution. No. Harmonic measure is defined by the bounded positive evaluation functional on continuous boundary data. A rough boundary may have no classical unit normal or usable arclength parametrization, and the measure need not have a classical normal-derivative density. General reciprocity remains valid by exhaustion, but it does not supply the missing boundary differential structure. Formula (13) belongs to the smooth-boundary statement in which it was proved.

Chapter 18

Explicit kernels and the topology of Green levels

18.1 The disk: a pole moved by a conformal automorphism

Proposition 18.1 (Disk kernel)**.** *For $z, p \in \mathbb{D}$, $z \neq p$,*

$$G_{\mathbb{D}}(z,p) = \log \left| \frac{1-\overline{p}z}{z-p} \right|. \tag{15}$$

Its boundary flux is

$$-\partial_{n_\zeta} G_{\mathbb{D}}(\zeta,p) = \frac{1-|p|^2}{|\zeta-p|^2}, \qquad |\zeta| = 1. \tag{16}$$

Proof. The disk automorphism $\phi_p(z) = (z-p)/(1-\overline{p}z)$ has its only zero at p, has nonzero derivative throughout the disk, and has modulus one on the circle. Pulling back $-\log|w|$ gives (15). Positivity can also be checked directly from

$$|1-\overline{p}z|^2 - |z-p|^2 = (1-|p|^2)(1-|z|^2) > 0.$$

For the normal derivative put $z = r\zeta$ and differentiate the two logarithms. Using $\partial_r \log|a(r)| = \mathrm{Re}(a'(r)/a(r))$ gives

$$\partial_r G_{\mathbb{D}}(r\zeta,p) = \mathrm{Re}\left(\frac{-\overline{p}\zeta}{1-r\overline{p}\zeta} - \frac{\zeta}{r\zeta-p} \right).$$

At $r = 1$, bringing the fractions to a common denominator, or multiplying by $\overline{\zeta}$, reduces this expression to $-(1-|p|^2)/|\zeta-p|^2$. Thus (13) becomes the Poisson formula with density $P(p,\zeta)/(2\pi)$ relative to $d\theta$. □

The equipotential $G = t$ is the inverse image of $|w| = e^{-t}$ under ϕ_p. If p is real, its full parametrization is

$$z(t,\theta) = \frac{p + e^{-t}e^{i\theta}}{1 + pe^{-t}e^{i\theta}}. \tag{17}$$

Keeping θ fixed gives a Green ray, with a parametrization that need not have constant Euclidean speed. Orthogonality follows either from conformality or from $\nabla G \cdot J\nabla G = 0$, where J rotates by $\pi/2$. These curves are circles or circular arcs. Their crowding near the boundary point closest to p is measured by the same Poisson density that governs harmonic measure.

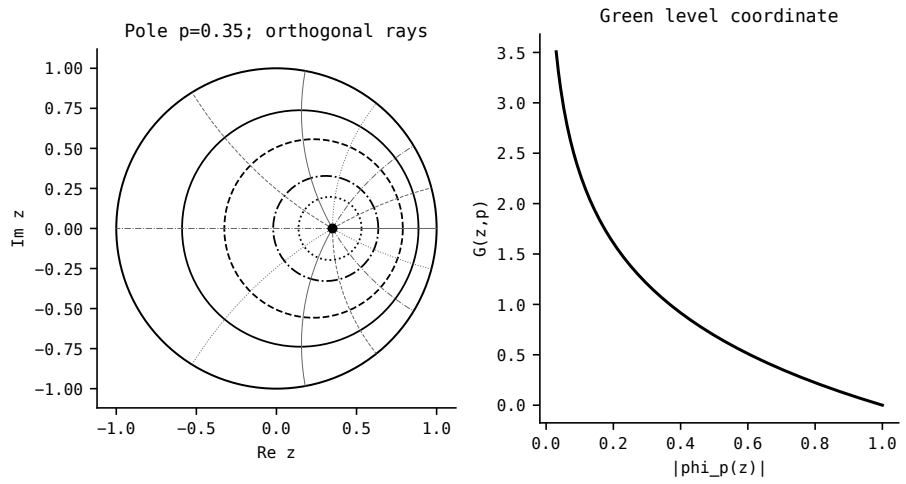


Figure 18.1: Exact Green equipotentials and rays in the unit disk with pole $p = 0.35$. The heavier level curves use (17), distinguished by line pattern; thin rays keep its angular coordinate fixed. The right panel shows $G = -\log|\phi_p|$. One disk automorphism transports the concentric circles and radial segments.

18.2 An annulus as a periodic cylinder

Let $A_L = \{e^{-L} < |z| < 1\}$, $L > 0$, and use coordinates

$$z = e^{-x+i\theta}, \quad 0 < x < L, \quad \theta \in \mathbb{R}/(2\pi\mathbb{Z}).$$

Although this coordinate reverses the orientation if read as $x + i\theta$, it is conformal as a metric coordinate: the Euclidean line element is $e^{-x}\sqrt{dx^2 + d\theta^2}$. Harmonicity is therefore equivalent to $u_{xx} + u_{\theta\theta} = 0$. Fix a pole (x_0, θ_0) and write $\vartheta = \theta - \theta_0$, $m = \min(x, x_0)$, $M = \max(x, x_0)$.

Proposition 18.2 (Annular Fourier kernel)**.** *The annular Green kernel is*

$$G_{A_L}(x, \theta; x_0, \theta_0) = \frac{m(L-M)}{L} + \sum_{n=1}^{\infty} \frac{2\sinh(nm)\sinh(n(L-M))}{n\sinh(nL)} \cos(n\vartheta). \tag{18}$$

The sum converges exponentially on compact sets with $x \neq x_0$. On the line $x = x_0$ its elementary logarithmic summation gives the continuous value away from the pole, and near the pole it is a logarithmic singularity plus a smooth harmonic function. In particular (18) is interpreted by that continuous off-pole representative, not by an assertion of absolute convergence at the pole.

Proof. The periodic delta has Fourier expansion

$$2\pi\delta_{\theta_0} = 1 + 2\sum_{n\geq 1} \cos(n\vartheta)$$

in distributions. Thus the zeroth mode g_0 must satisfy $-g_0'' = \delta_{x_0}$ with zero endpoint values. The piecewise linear solution is $m(L-M)/L$: it is continuous, and its derivative drops by one at x_0. For $n \geq 1$, a mode a_n satisfies

$$-a_n'' + n^2 a_n = 2\delta_{x_0}, \qquad a_n(0) = a_n(L) = 0.$$

On either side of x_0 the endpoint condition chooses a multiple of $\sinh(nx)$ or $\sinh(n(L-x))$. Continuity and the derivative jump $a_n'(x_0+) - a_n'(x_0-) = -2$ determine exactly the coefficient in (18).

To establish the meaning of the series, first rewrite that coefficient without exponentially large intermediate quantities:

$$a_n(x) = \frac{e^{-n|x-x_0|}}{n} \frac{(1-e^{-2nm})(1-e^{-2n(L-M)})}{1-e^{-2nL}}. \tag{19}$$

Away from $x = x_0$ it and all differentiated series have exponential decay on compact sets. Near the pole choose a small closed rectangle contained in $0 < x < L$. On it, subtract $e^{-n|x-x_0|}/n$ from a_n. The difference, including each one-sided derivative of any fixed order, decays exponentially in n; moreover the derivative jumps cancel. The remaining explicit series satisfies

$$-\frac{|t|}{2} + \sum_{n\geq 1} \frac{e^{-n|t|}}{n} \cos(n\vartheta) = -\frac{1}{2} \log\bigl(2(\cosh t - \cos\vartheta)\bigr), \qquad t = x - x_0, \tag{20}$$

initially for $t \neq 0$, by $-\log(1-w) = \sum_{n\geq 1} w^n/n$. The right side continues off $(t,\vartheta) = (0,0)$. Because

$$2(\cosh t - \cos\vartheta) = t^2 + \vartheta^2 + O((t^2+\vartheta^2)^2),$$

it has singularity $-\log\sqrt{t^2+\vartheta^2} + O(1)$. The combination $g_0 + |t|/2$ is affine near x_0, since its derivative jump is zero. Hence the remainder after subtracting (20) is smooth and harmonic, and (18) has the required coefficient-one pole.

Near $x = 0$, restrict first to $x < x_0/2$. Formula (19) is bounded by a summable multiple of $e^{-nx_0/2}/n$, and every coefficient tends to zero as $x \to 0$. Dominated convergence gives the zero boundary value, uniformly in angle. The argument at L is identical. We have now verified harmonicity, both boundary conditions, and the local singularity. Uniqueness identifies the sum with the Green kernel; its positivity is a theorem about the full sum, not a property of every cosine term. □

18.2.1 Subtracting the singular series before estimating derivatives

The exponential remainder assertion in the proof has an explicit coefficient form. Let $t = x - x_0$, and define $R_n(x) = a_n(x) - e^{-n|t|}/n$. Expanding the numerator of (19) gives the identity

$$R_n(x) = \frac{-e^{-n(x+x_0)} - e^{-n(2L-x-x_0)} + 2e^{-2nL}\cosh(nt)}{n(1-e^{-2nL})}.$$

Unlike the two terms from which it was formed, this expression is smooth across $x = x_0$. In particular the derivative jumps cancel algebraically, not only after an infinite summation. Suppose $x, x_0 \in [d, L-d]$ with $0 < d < L/2$. Every exponential in the expanded numerator has an exponent distance at least $2d$. For each integer $j \geq 0$,

$$|\partial_x^j R_n(x)| \leq \frac{4n^{j-1}e^{-2nd}}{1-e^{-2L}}.$$

An angular derivative of order ℓ contributes at most another factor n^ℓ. Thus every mixed differentiated remainder series is uniformly convergent on this closed strip, because $\sum n^{j+\ell-1}e^{-2nd} < \infty$. Finite partial sums solve the homogeneous equation there, and convergence of the second derivatives passes that equation to the limit. This verifies smooth harmonicity of the remainder by operations with summable bounds.

For a usable finite truncation, put $Q = e^{-2d}$. The value remainder beyond order $N \geq 0$ satisfies

$$\left|\sum_{n>N} R_n(x)\cos(n\vartheta)\right| \leq \frac{4Q^{N+1}}{(N+1)(1-Q)(1-e^{-2L})}.$$

Each first partial derivative has the corresponding upper bound $4Q^{N+1}/((1-Q)(1-e^{-2L}))$. These estimates stay finite on the pole latitude $x = x_0$ because the explicit logarithmic term (20) has already been removed. They do not bound the unsubtracted Green function at its pole; that function diverges. The order of the operations matters: identify the singular model, subtract it exactly, then estimate the smooth series.

Finally, the zero Fourier mode can be tracked without any limiting argument. The piecewise affine function $g_0(x) + |x - x_0|/2$ has derivative $(L - x_0)/L - 1/2$ on both sides of x_0. Hence it is one affine function across the pole latitude. Combining it with (20) and the remainder series accounts for all modes, including the constant mode that carries the net source flux.

Integrating (18) in the angular variable removes every nonzero mode:

$$\frac{1}{2\pi}\int_0^{2\pi} G_{A_L}(x, \theta; x_0, \theta_0)\, d\theta = \frac{m(L-M)}{L}. \tag{21}$$

The harmonic measures of the inner and outer circles are respectively x/L and $1 - x/L$. Accordingly, a source at x_0 sends fractions x_0/L and $1 - x_0/L$ of its total outward flux to those circles. These fractions can be obtained either from the harmonic-measure representation or by differentiating the zeroth mode at $x = 0, L$.

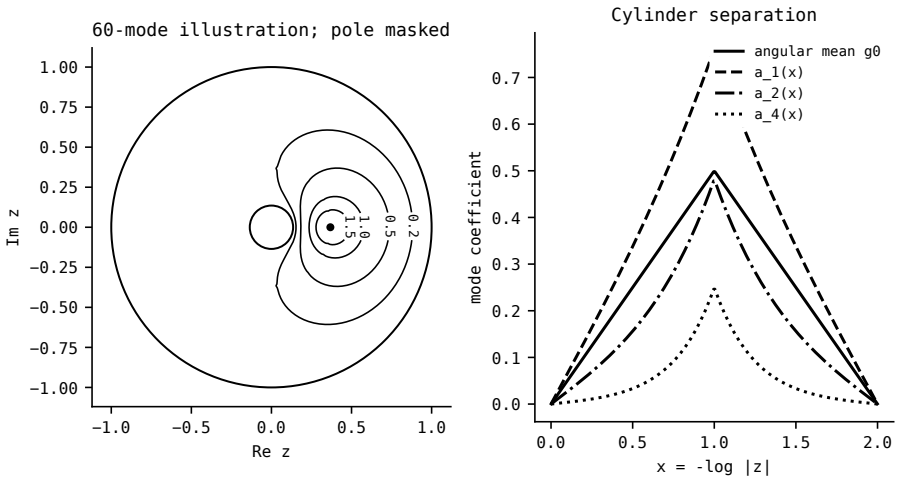


Figure 18.2: A Green kernel on the annulus $e^{-2} < |z| < 1$, with pole $(x_0, \theta_0) = (1, 0)$ in periodic-cylinder coordinates. The left panel uses 60 modes of (18), with the pole masked; the right panel shows its angular mean and the coefficients a_1, a_2, a_4 as functions of x. The analytic formula and its convergence proof, rather than the plotted samples, establish the boundary values and singularity.

18.3 The critical-point equivalence without a connectivity cutoff

Lemma 18.3 (Periods after removing one point)**.** *If a noncompact surface domain U is simply connected and $p \in U$, then every period of a closed one-form on $U \setminus \{p\}$ is an integer multiple of its period on a small positively oriented coordinate circle about p.*

Proof. Take a closed piecewise smooth curve γ in the punctured domain. In U it is null-homotopic. We use the classical relative smooth approximation and transversality theorem for a map of a compact disk into a smooth surface, with its boundary fixed away from p. Subdivide the resulting null-homotopy into finitely many coordinate pieces and perturb its interior to meet the point p only at finitely many transverse preimages. Delete small disks around those preimages. The boundary

of the remaining parameter surface consists of γ and finitely many small curves that map, with orientation signs, around p. Stokes' theorem for the pulled-back closed form says that the period on γ equals the signed sum of these small-circle periods. Shrinking the coordinate circles does not change their period, again by Stokes on annuli. Thus the sum is an integer multiple of one circle period. Equivalently, this is the elementary homology computation $H_1(U \setminus \{p\}; \mathbb{Z}) \simeq \mathbb{Z}$ for a simply connected surface. The finite subdivision is why no finite-connectivity assumption on the boundary of U enters this calculation. □

Theorem 18.4 (Critical points detect nonsimple connectivity)**.** *Let U be any regular Green domain on a Riemann surface and let $p \in U$. Then*

$$G_U(\cdot, p) \text{ has no critical point on } U \setminus \{p\} \quad \Longleftrightarrow \quad U \text{ is simply connected.} \tag{22}$$

Here a critical point means $dG = 0$ in a local coordinate. The assertion allows arbitrary boundary connectivity and does not assume that U is a planar domain.

Proof. First suppose that U is simply connected. A local harmonic conjugate H of G satisfies $dH = d^cG$. At the pole,

$$G = -\log|w| + h, \qquad d^cG = -d\arg w + d^ch,$$

so its period around a small positive circle is -2π. To make the forthcoming path-independence claim explicit, fix a base point and define a local value of $H(z)$ by integrating d^cG along a path to z. Two paths change this value by a closed-curve integral. Once that integral is known to be $2\pi\ell$, $\ell \in \mathbb{Z}$, their exponentials differ only by $e^{-2\pi i\ell} = 1$. On each simply connected coordinate disk avoiding the pole, the primitive satisfies the Cauchy–Riemann equations with G, so the exponential is holomorphic there. Path independence then makes these local expressions agree on overlaps. This is the needed global conclusion; the conjugate itself does not have to acquire a global real-valued branch.

Lemma 18.3 shows that all periods are multiples of 2π. It follows that

$$F = \exp(-G - iH) \tag{23}$$

is a single-valued holomorphic function on the punctured domain, even though H itself need not be single-valued there. Locally, if k is a holomorphic function with real part h, then $F = cwe^{-k(w)}$ for a constant c of modulus one. Thus F extends across p with one simple zero and has no other zero. Also $|F| = e^{-G} < 1$ and $|F| \to 1$ at the boundary.

The map $F : U \to \mathbb{D}$ is proper. Indeed, the inverse image of a compact set $K \subset \mathbb{D}$ lies in a superlevel $G \geq -\log r$, where $r < 1$ bounds $|w|$ on K, with the pole included. This superlevel is compact in U. The image of a nonconstant holomorphic map is open; properness makes it closed in $\mathbb{D}$. Hence the image is all of $\mathbb{D}$. To prove injectivity, fix $a \in \mathbb{D}$ and choose a regular level $t > 0$ with $|a| < e^{-t}$. On the compact bordered region $U_t = \{G > t\}$,

$$|a| < |F| = e^{-t} \quad \text{on } \partial U_t.$$

For completeness, the zero count can also be read through a boundary homotopy. For $0 \leq s \leq 1$, the functions $F - sa$ never vanish on ∂U_t, since $|F - sa| \geq e^{-t} - |a| > 0$ there. Consequently the integer

$$\frac{1}{2\pi i}\int_{\partial U_t} \frac{d(F - sa)}{F - sa}$$

is independent of s. At $s = 0$ its value is one because F has one simple zero and no poles; at $s = 1$ it counts the zeros of $F - a$. The surface argument principle applies to the finitely many boundary curves with their induced orientations, including any inner curves. Moreover, $F(z) = a$

implies $G(z) = -\log|a| > t$ when $a \neq 0$, while for $a = 0$ the unique preimage is p. Thus there is no additional fiber outside the counted region. The argument principle, or its Rouché form on a bordered surface, shows that $F - a$ has as many zeros in U_t, counted with multiplicity, as F: exactly one. Every preimage of a lies in U_t. Thus each fiber contains one simple point. The local inverse theorem makes F a conformal isomorphism. Consequently $dG = -d\log|F|$ never vanishes away from p, since $F' \neq 0$ and $F \neq 0$ there.

Conversely suppose that dG never vanishes away from p. Choose any smooth compatible metric and define

$$X = \frac{\nabla G}{|\nabla G|^2} \quad \text{on } U \setminus \{p\}. \tag{24}$$

Along its integral curves, $dG/ds = 1$. This vector field need not be globally bounded, and no global boundedness is needed. For every $0 < a < b < \infty$, the slab

$$K_{a,b} = \{z \in U : a \leq G(z) \leq b\}$$

is compact and avoids p. Its compactness follows from the zero boundary limit at one end and the divergent pole at the other. On a neighborhood of this slab, X is smooth and bounded in finitely many coordinate charts. Standard local existence and uniqueness for ODEs therefore continues each flow line for the entire finite time needed to move between any two levels in $[a, b]$: an earlier maximal endpoint would have a cluster point in the compact slab where the local theorem extends the solution.

More explicitly, choose a slightly larger slab $K_{a/2,b+1}$. Continuity and the assumption of no critical points imply

$$c_{a,b} := \min_{K_{a/2,b+1}} |\nabla G| > 0, \qquad |X| = |\nabla G|^{-1} \leq c_{a,b}^{-1}.$$

In finitely many relatively compact coordinate charts meeting this slab, the coefficients of X and their first derivatives have finite bounds. Suppose a flow starting at level $t_0 \in [a, b]$ had a finite maximal time $s_* < b - t_0$. The identity $G(\Phi_s) = t_0 + s$ keeps it in the compact slab. Its bounded speed makes it a Cauchy curve as $s \uparrow s_*$ with respect to a compatible Riemannian distance on a compact neighborhood. Its limit lies inside that neighborhood, where a coordinate existence theorem extends the curve past s_*. This contradiction proves forward continuation; reversing time proves the backward assertion. The positive lower bound is local to the chosen finite slab and is never asserted uniformly as $a \downarrow 0$ or $b \uparrow \infty$.

For large T, the level $G = T$ is one circle. To see this with no assumption about other parts of U, first choose a small pole chart. The function is bounded on the complement of a smaller chart, including near the outer boundary. Thus large superlevels lie in the chart. There the holomorphic coordinate $W = we^{-k(w)}$ has nonzero derivative at zero and gives $G = -\log|W|$. Large levels are therefore the circles $|W| = e^{-T}$.

Fix one such level C_T. Flow by (24) gives a bijection

$$C_T \times (0, \infty) \longrightarrow U \setminus \{p\}, \qquad (\xi, t) \longmapsto \Phi_{t-T}(\xi). \tag{25}$$

Existence follows from continuation across compact slabs, injectivity from uniqueness, and surjectivity by flowing any point to level T. Smooth dependence on initial conditions gives a diffeomorphism. Identify C_T with a circle, send level t to radius e^{-t}, and send p to the origin. The resulting map $U \to \mathbb{D}$ is a homeomorphism: its continuity at p follows because high superlevels shrink into every pole neighborhood; the inverse has the same property. A disk is simply connected. This proves the reverse implication and the complete stated equivalence. □

18.4 The local shape of a critical level

The topology theorem concerns existence of a critical point. Its local geometry can be described more sharply. At a critical point $q \neq p$, choose a coordinate $w(q) = 0$ and a holomorphic function F with $\operatorname{Re} F = G$. Since G is not constant, its Taylor series has a first nonzero nonconstant term:

$$F(w) - F(0) = aw^k(1 + O(w)), \qquad a \neq 0, \quad k \geq 2.$$

On a sufficiently small disk, the nonvanishing analytic factor $a(1 + O(w))$ has an analytic kth root. This follows by choosing its holomorphic logarithm and exponentiating one kth of it. The coordinate

$$W = w\big(a(1 + O(w))\big)^{1/k}$$

has nonzero derivative at zero and puts the function into the exact form

$$G - G(q) = \operatorname{Re}(W^k). \tag{22a}$$

Thus the critical level consists of $2k$ arcs meeting at equal angles π/k in this conformal coordinate. The sectors between them alternate between $G > G(q)$ and $G < G(q)$. Every nonconstant harmonic critical point is therefore a multisector saddle; no isolated local maximum or minimum has been hidden in the level-flow discussion.

The zero order of $G_x - iG_y$ is $k - 1$. For a small loop about the point, the complex gradient $G_x + iG_y$ turns $-(k - 1)$ times, since it is the complex conjugate of a holomorphic function with a zero of that order. This explains the multiplicities in the following count. A degenerate saddle with $k > 2$ contributes more than one to the critical-point number, even though it occupies only one spatial location. A numerical root finder returning a list of distinct locations would need multiplicity information before its list could be compared with that theorem.

18.4.1 Reading trajectories through an exact model

For the disk, (17) provides the level flow in closed form. Let $0 \leq p < 1$ be real, set $r = e^{-t}$, and hold θ fixed. Direct differentiation gives

$$\partial_t z(t, \theta) = -\frac{(1 - p^2)e^{-t}e^{i\theta}}{(1 + pe^{-t}e^{i\theta})^2}, \qquad |\partial_t z| = \frac{(1 - p^2)e^{-t}}{|1 + pe^{-t}e^{i\theta}|^2}.$$

Since the disk automorphism takes this curve to $e^{-t}e^{i\theta}$, the curve is orthogonal to the Green levels and $dG/dt = 1$. It is therefore precisely the integral curve of (24), in the Euclidean metric. The triangle inequality yields

$$\frac{(1 - p^2)e^{-t}}{(1 + pe^{-t})^2} \leq |\partial_t z| \leq \frac{(1 - p^2)e^{-t}}{(1 - pe^{-t})^2}.$$

Each finite interval of levels has finite positive speed bounds; the speed tends to zero as $t \to \infty$, where the curve reaches the pole only in the limiting level parameter. The divergence of G does not mean that the Euclidean speed must diverge.

The critical models behave differently. For $g_k(W) = \operatorname{Re}(W^k)$, $k \geq 2$,

$$|\nabla g_k| = k|W|^{k-1}, \qquad |\nabla g_k/|\nabla g_k|^2| = \frac{1}{k|W|^{k-1}}.$$

The normalized level-flow field is undefined at the saddle and its speed diverges when it approaches that point. Its zero level consists of rays with angles $(\pi/2 + j\pi)/k$, $j = 0, \ldots, 2k - 1$. These explicit formulas show exactly why a proof that crosses critical levels cannot reuse the positive lower bound $c_{a,b}$ from the noncritical case. The models illustrate a local obstruction, not a claim that an arbitrary Green function equals a monomial globally.

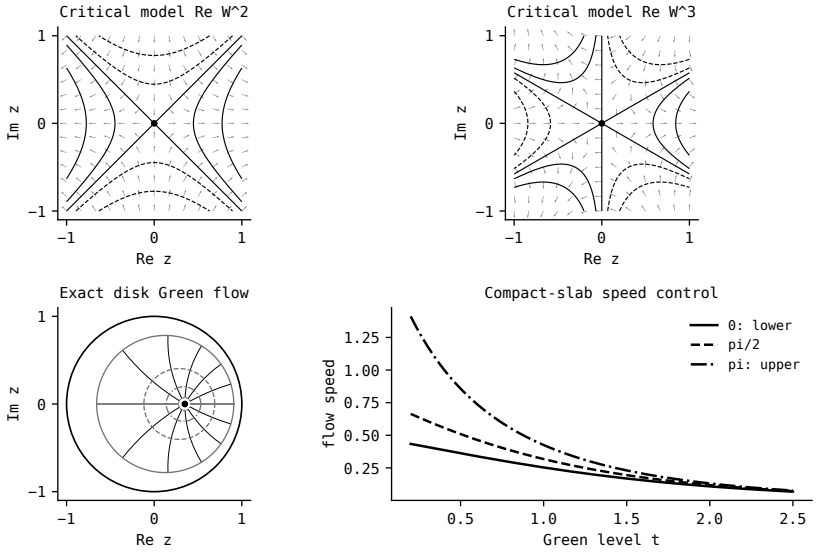


Figure 18.3: Exact local and global flow models. The first panels display the level and gradient geometry of $\mathrm{Re}(W^2)$ and $\mathrm{Re}(W^3)$, including their four and six zero-level rays. The disk panel uses (17) with $p = 0.35$ and fixed angular parameters. The speed panel compares $|\partial_t z|$ at $\theta = 0, \pi/2, \pi$ with the two analytic envelopes in this subsection. The monomial portraits are local saddle models; the disk trajectories are exact Green-flow curves. The lower and upper speed envelopes are attained at $\theta = 0$ and $\theta = \pi$, respectively, so those envelopes coincide with the two extreme curves.

18.5 A numerical critical-point count in a narrower class

Proposition 18.5 (Planar finite-boundary count)**.** *Let $U \subset \mathbb{C}$ be bounded, with boundary consisting of $m \geq 1$ disjoint analytic Jordan curves. The Green function has exactly $m-1$ critical points, counted with the orders of zeros of $G_x - iG_y$.*

Proof. The meromorphic function $f = G_x - iG_y$ has a simple pole at p and no other poles. Along the boundary it has no zero. For the latter assertion, the inward derivative of G is strictly positive: at a boundary point put a small interior tangent disk and compare G on an annular collar of that disk with a positive multiple of $\log(R/|z-c|)$. The multiple can be chosen because G has a positive minimum on the inner circle. At the contact point the comparison gives a strictly positive inward derivative. Analytic boundary reflection provides the differentiability used here.

Orient every boundary component with U on its left and let τ be its complex unit tangent. The inward normal is $i\tau$, so

$$G_x + iG_y = \kappa i\tau, \quad f = -i\kappa\overline{\tau}, \quad \kappa > 0.$$

Consequently the winding of f on a component is minus the turning number of its tangent. The outer Jordan curve has turning number $+1$; each of the $m-1$ inner curves has turning number -1. The total winding of f is therefore $m-2$. By the argument principle this equals its number of zeros minus its one pole, giving $N - 1 = m - 2$ and $N = m - 1$. □

The count requires the stated planar topology and finite smooth boundary. It is not a count for arbitrary surface domains. The equivalence (22) was proved independently and retains the broader scope of the source's Exercise 10.24.

18.6 Solved investigations

Investigation 18.6 (The regular part of the disk kernel)**.** Find $\lim_{z\to p}(G_{\mathbb{D}}(z,p)+\log|z-p|)$.

Solution. Formula (15) gives $\log(1-|p|^2)$. Thus the regular part decreases to $-\infty$ as the pole approaches the grounded circle. The associated conformal radius in the coordinate z is

$$r_{\mathbb{D}}(p)=e^{\log(1-|p|^2)}=1-|p|^2.$$

For D_R, scaling gives $r_{D_R}(p)=(R^2-|p|^2)/R$.

Investigation 18.7 (Two boundaries divide one source)**.** For the annulus $e^{-L}<|z|<1$, find the inner-boundary harmonic measure at a point of radius r.

Solution. The radial harmonic function with values one at e^{-L} and zero at 1 is

$$\omega^z(\{|\zeta|=e^{-L}\})=\frac{-\log r}{L}.$$

Its complementary outer mass is $1+\log r/L$. At $r=e^{-L/2}$ the two masses are equal, even though the two circumferences differ. Flux fractions depend on logarithmic distance in the cylinder coordinate.

Investigation 18.8 (One annular saddle)**.** How many critical points does the annular Green function have, with multiplicity? Must their type be a maximum or minimum?

Solution. The boundary count has $m=2$, so there is one critical point of multiplicity one. A nonconstant harmonic function has no interior maximum or minimum. Locally at a simple critical point it is

$$G(z)=G(z_0)+\operatorname{Re}(a(z-z_0)^2)+O(|z-z_0|^3),\qquad a\neq 0,$$

so the level set through the point has four alternating sectors. It is a saddle. A finite contour plot can suggest its location, but the count comes from the argument principle.

Investigation 18.9 (Why a puncture breaks the topology test)**.** Compare (22) with the restriction of $G_{\mathbb{D}}(z,p)$ to $\mathbb{D}\setminus\{0\}$, where $p\neq 0$.

Solution. The restricted kernel has no critical point, since its disk automorphism has nonzero derivative. The punctured domain is not simply connected. There is no contradiction: the restricted function has a positive value at the missing point, so it fails the zero boundary limit there. Its positive-level slabs can approach that deleted point and are not compact in the punctured domain. The precise step that fails in the flow proof is therefore identifiable.

Investigation 18.10 (A bound for annular modes away from the pole latitude)**.** Suppose $|x-x_0|\geq\delta>0$. Bound the tail in (18) after mode N.

Solution. By (19),

$$|a_n(x)|\leq\frac{e^{-n\delta}}{n(1-e^{-2L})}.$$

Hence, uniformly in angle,

$$\left|\sum_{n>N}a_n(x)\cos(n\vartheta)\right|\leq\frac{e^{-(N+1)\delta}}{(N+1)(1-e^{-2L})(1-e^{-\delta})}.$$

This certificate degenerates as $\delta\downarrow 0$. On the pole latitude one should subtract the explicit logarithm (20) before estimating a remainder; applying this bound with $\delta=0$ would be meaningless.

Chapter 19

Dirichlet energy and condenser capacity

19.1 An exact variational identity

For a bounded planar domain V and a real function u with square integrable gradient, write

$$\mathcal{E}_V(u) = \int_V |\nabla u|^2 \, dA, \qquad \mathcal{E}_V(u, v) = \int_V \nabla u \cdot \nabla v \, dA. \tag{26}$$

The factor $1/2$ used for physical stored energy is not included in this mathematical definition. Initially take V to have smooth boundary and all competing functions to be C^1 on its closure; the harmonic minimizer is C^2 in the interior and sufficiently smooth up to the boundary for integration by parts. These hypotheses give a classical proof before introducing any Sobolev completion.

Theorem 19.1 (Dirichlet's principle as a proved identity)**.** *If h is harmonic with prescribed boundary data g, then every admissible u with the same boundary data satisfies*

$$\mathcal{E}_V(u) = \mathcal{E}_V(h) + \mathcal{E}_V(u - h). \tag{27}$$

Hence h is the unique energy minimizer. Conversely, a C^2 minimizer among these competitors is harmonic.

Proof. Set $v = u - h$. It vanishes on the boundary, and expanding the square gives

$$\mathcal{E}_V(h + v) = \mathcal{E}_V(h) + 2\mathcal{E}_V(h, v) + \mathcal{E}_V(v).$$

Green's first identity gives

$$\mathcal{E}_V(h, v) = \int_{\partial V} v \partial_n h \, ds - \int_V v \Delta h \, dA = 0,$$

which proves (27). Equality of the two energies forces $\int_V |\nabla v|^2 = 0$. Continuity of the gradient then gives $\nabla v = 0$ everywhere. Connectedness makes v constant, and its zero boundary value makes that constant zero.

Conversely, let h be a minimizer and let ϕ be any smooth compactly supported test function. The quadratic polynomial

$$\mathcal{E}_V(h + t\phi) - \mathcal{E}_V(h) = 2t \int_V \nabla h \cdot \nabla \phi \, dA + t^2 \mathcal{E}_V(\phi)$$

is nonnegative for every real t. Its linear coefficient must vanish; otherwise taking sufficiently small t of the opposite sign gives a negative value. Integration by parts yields $\int_V \phi \Delta h = 0$ for every ϕ. If the continuous function Δh were positive or negative at some point, a nonnegative test function supported in a small neighborhood of that point would contradict the integral equality. Thus $\Delta h = 0$ pointwise. □

Existence has not been deduced from the phrase "minimize the energy". It was supplied earlier by the Dirichlet theorem. Formula (27) then proves minimization. To construct a minimizer directly in a completed function space would require compactness, trace, and regularity theorems; none is being silently substituted for that earlier existence proof.

Proposition 19.2 (Conformal invariance of energy)**.** *If $F : V \to W$ is conformal and $u \in C^1(W)$ has finite Dirichlet energy on W, then $\mathcal{E}_V(u \circ F) = \mathcal{E}_W(u)$.*

Proof. In real coordinates $DF = |F'|R$, where R is a rotation. The chain rule and change of variables give

$$|\nabla(u \circ F)(z)|^2 = |F'(z)|^2 |\nabla u(F(z))|^2, \qquad dA(F(z)) = |F'(z)|^2 dA(z).$$

Integrating cancels the two factors. Exhaustion by compact subsets and monotone convergence extend the calculation to a finite-energy function when the domains are not compact. The same cancellation in (11) shows that Dirichlet energy is intrinsically defined on a Riemann surface. □

19.2 Two electrodes and a reciprocal flux problem

Let V have two disjoint analytic boundary components E_1, E_0, and let w solve

$$\Delta w = 0 \text{ in } V, \qquad w|_{E_1} = 1, \quad w|_{E_0} = 0.$$

The condenser capacity, in the present normalization, is

$$C = \operatorname{Cap}(V; E_1, E_0) = \mathcal{E}_V(w) = \inf_{u|_{E_1}=1,\, u|_{E_0}=0} \mathcal{E}_V(u). \tag{28}$$

The exact competitor class is the classical $C^1(\overline{V})$ class in the preceding section with the specified pointwise boundary values. The harmonic solution belongs to this class: near either analytic boundary component, subtract its constant boundary value and apply zero-boundary harmonic reflection. No completed function space or general trace theorem is asserted in this definition. The classical identity proves the value for every admissible competitor. The capacity is strictly positive: zero energy would make w constant, which is incompatible with its two boundary values.

Green's identity gives its flux interpretation:

$$C = \int_{E_1} \partial_n w \, ds, \qquad \int_{E_0} \partial_n w \, ds = -C. \tag{29}$$

Here n points out of the field region V, so it points into the inner electrode when E_1 is the inner boundary. The first flux is positive; the second is negative. The vector field $-\nabla w$ flows from the high-potential electrode toward the low-potential one, and consequently has outward flux $+C$ through E_0.

Theorem 19.3 (Unit-flow duality)**.** *Among C^1 vector fields J on $\overline{V}$ satisfying*

$$\operatorname{div} J = 0, \qquad \int_{E_0} J \cdot n \, ds = 1,$$

the minimum of $\int_V |J|^2 dA$ is $1/C$. It is attained uniquely by $J_* = -\nabla w / C$.

Proof. The divergence theorem forces the flux on E_1 to be -1. Since $w = 1$ there and zero on E_0,

$$\int_V J \cdot \nabla w \, dA = \int_{\partial V} wJ \cdot n \, ds - \int_V w \operatorname{div} J \, dA = -1.$$

Cauchy–Schwarz yields

$$1 \le \left(\int_V |J|^2 dA \right) \left(\int_V |\nabla w|^2 dA \right) = C \int_V |J|^2 dA. \tag{30}$$

The field J_* is divergence free, has unit outer flux by (29), and has energy $C/C^2 = 1/C$, so the bound is attained. More precisely, writing $J = J_* + K$, the flux of K on both electrodes is zero and $\operatorname{div} K = 0$. The same integration by parts gives $\int_V K \cdot \nabla w = 0$. Therefore

$$\int_V |J|^2 dA = \frac{1}{C} + \int_V |K|^2 dA.$$

This identity proves uniqueness and quantifies the excess dissipation of every alternative unit flow. □

19.2.1 A computable gap from an admissible potential and flow

The two minimization principles can be combined without knowing their minimizers. Let $u \in C^1(\overline{V})$ have boundary values one and zero, and let J be a classical divergence-free unit flow as in Theorem 19.3. Set

$$A = \int_V |\nabla u|^2 dA, \qquad B = \int_V |J|^2 dA.$$

Neither integral vanishes: the potential has different boundary values, and the flow has nonzero total flux. Integration by parts, using the actual boundary values of u rather than its interior equation, gives

$$\int_V J \cdot \nabla u \, dA = \int_{E_1} J \cdot n \, ds = -1.$$

Consequently the nonnegative square has the exact value

$$\int_V \left| \nabla u + \frac{J}{B} \right|^2 dA = A - \frac{2}{B} + \frac{B}{B^2} = A - \frac{1}{B}.$$

Together with the two principles this proves

$$\frac{1}{B} \le C \le A, \qquad 0 \le \|\nabla(u - w)\|_2^2 = A - C \le A - \frac{1}{B}.$$

The interval width thus controls the potential's squared gradient error as well as the capacity uncertainty. If $AB - 1 \le \delta$, the upper endpoint is at most $1 + \delta$ times the lower endpoint, so this is a relative certificate independent of the scale of the capacity.

The corresponding flow identity is $\|J + \nabla w/C\|_2^2 = B - 1/C$. Since $C \le A$,

$$0 \le B - \frac{1}{C} \le B - \frac{1}{A}.$$

Thus both trial objects approach their respective optimizers when the bracket closes. These conclusions concern energy norms. They do not assert a maximum-norm bound on the trial potential or pointwise control of the flow.

The admissibility hypotheses are necessary to the algebra. If a proposed field has divergence r and incorrect electrode fluxes, the integration by parts instead contains $-\int_V ur$ and the boundary flux error. Without certified bounds on those extra terms, the displayed square need not equal $A - 1/B$, and the lower capacity bound has not been established. A solver residual is a separate datum, not a license to discard these terms.

19.3 The annulus and the capacity–modulus reciprocal

For $V = \{a < |z| < b\}$, $0 < a < b$, the unique condenser potential is

$$w(r) = \frac{\log(b/r)}{\log(b/a)}, \qquad w'(r) = -\frac{1}{r\log(b/a)}. \tag{31}$$

It is harmonic because $w'' + r^{-1}w' = 0$, and its boundary values are one and zero. Direct integration gives

$$C = 2\pi \int_a^b \frac{dr}{r\log^2(b/a)} = \frac{2\pi}{\log(b/a)}, \qquad \operatorname{mod}(V) = \frac{\log(b/a)}{2\pi} = C^{-1}. \tag{32}$$

The minimizing unit current is $J_*(r) = \mathbf{e}_r/(2\pi r)$, independent of the outer radius at a fixed physical position. Its dissipation is

$$\int_a^b \frac{1}{4\pi^2 r^2}\, 2\pi r\, dr = \frac{\log(b/a)}{2\pi}.$$

Capacity is dimensionless in this two-dimensional normalization. Scaling both radii by the same positive factor leaves the ratio, energy, and capacity unchanged. A physical permittivity or conductivity restores the appropriate units in the next chapter.

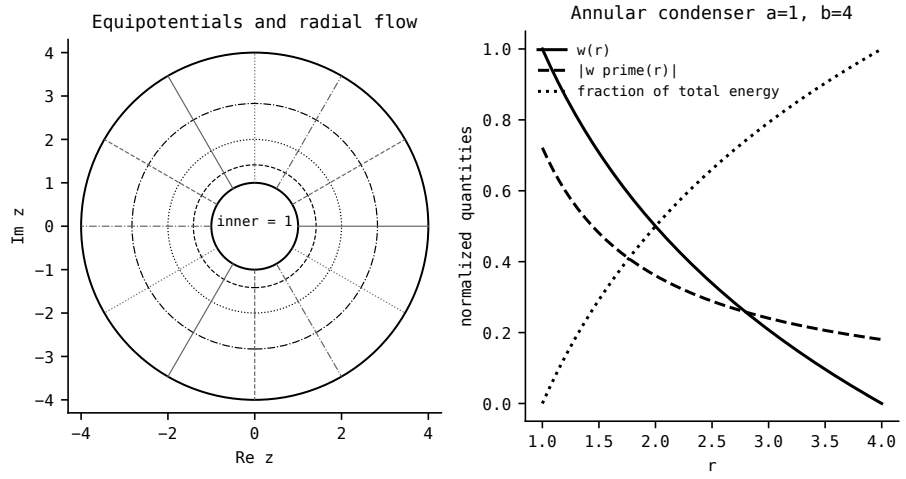


Figure 19.1: The annular condenser with radii $a = 1$ and $b = 4$: radial potential and flow geometry. The right panel shows $w(r)$, $|w'(r)|$, and the fraction $\log r/\log 4$ of total Dirichlet energy accumulated through radius r. The corresponding unnormalized energy is $2\pi \log r/(\log 4)^2$. Line patterns distinguish the quantities without color coding.

19.4 Capacity between arbitrary Green levels

The annular formula has an intrinsic counterpart that does not require the original domain to be an annulus. Let $G = G_U(\cdot, p)$ on a regular Green domain, and choose regular values $0 < a < b$. Set

$$W_{a,b} = \{a < G < b\}, \qquad w = \frac{G - a}{b - a}.$$

The closure of this field region is compact in the punctured domain. Its boundary consists of finitely many analytic curves, even when the original outer boundary has infinitely many

components. The normalized potential equals zero on $G = a$ and one on $G = b$, and is harmonic between them. The Dirichlet argument applies component by component if necessary.

For any regular value t, apply the divergence theorem to the superlevel $\{G > t\}$ after deleting a small pole disk. The small-circle flux is 2π with respect to the outward normal of the punctured region. Since G is harmonic on that region, the outer level must contribute -2π. On that level the outward normal points toward decreasing G, so

$$\int_{G=t} |\nabla G|\, ds = 2\pi. \tag{32a}$$

Now apply Green's first identity to G on $W_{a,b}$. The normal points toward increasing G on the upper boundary and toward decreasing G on the lower one. Hence

$$\int_{W_{a,b}} |\nabla G|^2 dA = b \int_{G=b} |\nabla G|\, ds - a \int_{G=a} |\nabla G|\, ds = 2\pi(b-a).$$

Therefore

$$\operatorname{Cap}(W_{a,b}; G = b, G = a) = \frac{2\pi}{b-a}. \tag{32b}$$

Critical points may occur inside the slab. They affect the arrangement of level curves but do not change this energy identity. The necessary regularity was imposed only on its two boundary levels.

In the disk with a central pole, $G = -\log r$, so $a < G < b$ is exactly $e^{-b} < r < e^{-a}$ and (32b) reduces to (32). The more general formula shows that a Green coordinate measures reciprocal capacity in equal increments. If $a < c < b$ are regular values, the reciprocal capacities add:

$$\frac{b-a}{2\pi} = \frac{c-a}{2\pi} + \frac{b-c}{2\pi}.$$

This series law follows from flux conservation and energy additivity; it does not require that every intermediate level have one component.

19.5 Several electrodes and a symmetric response matrix

Suppose now that ∂V consists of m analytic components $E_1, \dots, E_m$. Let w_j be harmonic with value one on E_j and zero on the others. Every choice of constant electrode potentials $a = (a_1, \dots, a_m)$ gives

$$u_a = \sum_{j=1}^{m} a_j w_j, \qquad C_{ij} = \int_V \nabla w_i \cdot \nabla w_j \, dA. \tag{33}$$

The matrix C is symmetric by its definition. Integration by parts and the boundary values identify its entries as fluxes:

$$C_{ij} = \int_{E_i} \partial_n w_j \, ds, \qquad q_i := \int_{E_i} \partial_n u_a \, ds = (Ca)_i. \tag{34}$$

Since $\sum_j w_j = 1$ by uniqueness, $C\mathbf{1} = 0$. Also

$$a^T C a = \mathcal{E}_V(u_a) \geq 0, \qquad a^T C a = 0 \Longleftrightarrow a_1 = \cdots = a_m. \tag{35}$$

Thus the sole null direction is a common potential shift, and the rank is $m - 1$. For $i \neq j$, w_j has minimum zero on E_i, so its outward normal derivative there is nonpositive. Hence $C_{ij} \leq 0$. The diagonal entries equal minus the sum of the other entries in their row and are positive when $m \geq 2$.

These statements are the continuum counterparts of a graph Laplacian's symmetry, nonpositive off-diagonal entries, zero row sums, and positive energy. The correspondence has been established by identities, not by declaring a discretization to be exact. Approximating this matrix by a mesh requires separate approximation and arithmetic bounds.

19.5.1 Fixing the additive constant and prescribing balanced flux

Assume $m \geq 2$. The rank statement can be made into an explicit solvability result. Ground electrode E_m by imposing $a_m = 0$, and let C_0 be the principal $(m-1) \times (m-1)$ submatrix of C. For any nonzero $b \in \mathbb{R}^{m-1}$, the vector $a = (b, 0)$ is not constant, so (35) gives

$$b^T C_0 b > 0.$$

In particular $C_0 b = 0$ implies $b = 0$; finite-dimensional linear algebra then makes C_0 invertible. Given electrode fluxes $q = (q_1, \ldots, q_m)$ satisfying $\sum_i q_i = 0$, define

$$b = C_0^{-1}(q_1, \ldots, q_{m-1})^T, \qquad a = (b, 0).$$

The first $m-1$ rows of $Ca = q$ hold by construction. Their sum and the zero row sums of C force the final row as well. Conversely any realizable q must be balanced because the total flux of a harmonic function is zero. This proves both necessity and sufficiency, with the ground fixing the only ambiguity.

Write $q_0 = (q_1, \ldots, q_{m-1})^T$. The resulting energy is

$$\mathcal{E}_V(u_a) = a^T q = q_0^T C_0^{-1} q_0.$$

Symmetry of C_0^{-1} expresses reciprocity for prescribed currents, just as symmetry of C expressed it for prescribed potentials. For perturbations, define the positive number

$$\gamma = \min_{|b|=1} b^T C_0 b > 0.$$

The minimum exists by compactness of the unit sphere; positivity follows because a zero minimum would contradict the strict inequality above. If $C_0 \delta b = \delta q_0$, Cauchy–Schwarz gives

$$\gamma |\delta b|^2 \leq \delta b^T C_0 \delta b = \delta b \cdot \delta q_0 \leq |\delta b|\, |\delta q_0|, \qquad |\delta b| \leq \gamma^{-1} |\delta q_0|.$$

For $\delta b = 0$ the last inequality is immediate; otherwise division by $|\delta b|$ is legitimate. This is a rigorous sensitivity formula conditional on a proved lower bound for γ. No numerical value of γ is supplied by merely knowing that the matrix is positive definite.

19.6 Solved investigations

Investigation 19.4 (An exact error norm)**.** If an approximate potential v has the correct boundary values, express its energy excess over the harmonic solution h.

Solution. Formula (27) gives

$$\mathcal{E}_V(v) - \mathcal{E}_V(h) = \|\nabla(v - h)\|^2_{L^2(V)}.$$

The excess is exactly the squared gradient error, not merely an upper bound. It does not by itself bound a pointwise error without an additional embedding or maximum-principle estimate. In two dimensions, a small gradient norm alone is not a universal sup-norm certificate.

Investigation 19.5 (Two concentric gaps in series)**.** Let $a < c < b$. Compare the annular resistances across (a, c) and (c, b) with the resistance across (a, b).

Solution. With resistance defined as reciprocal capacity,

$$R(a, b) = \frac{\log(b/a)}{2\pi} = \frac{\log(c/a)}{2\pi} + \frac{\log(b/c)}{2\pi} = R(a, c) + R(c, b).$$

The normalized radial unit current is the same in both regions, and the two energies add because the regions are disjoint. The interface circle is an equipotential, so this series decomposition is exact.

Investigation 19.6 (The two-electrode response matrix)**.** Determine (33) when $m = 2$ and the condenser capacity is C_0.

Solution. Since $w_2 = 1 - w_1$, the gradient of w_2 is the negative of the gradient of w_1. Therefore

$$C = C_0 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad a^T C a = C_0 (a_1 - a_2)^2.$$

The nonzero eigenvalue is $2C_0$, while a unit voltage difference produces flux C_0. Eigenvalue and two-terminal conductance should therefore not be conflated.

Investigation 19.7 (A uniform conductivity)**.** If the dissipation functional is $\sigma \int_V |\nabla u|^2 dA$ with constant $\sigma > 0$, find the two-terminal conductance and resistance.

Solution. The minimizing potential is unchanged, since multiplying an objective by a positive constant does not change its minimizer. The current density is $-\sigma \nabla u$, so a unit voltage difference produces current σC. Thus

$$\mathsf{G} = \sigma C, \qquad \mathsf{R} = (\sigma C)^{-1}.$$

For a spatially varying conductivity, the equation becomes $\operatorname{div}(\sigma(x)\nabla u) = 0$ and the unweighted harmonic solution generally changes; substituting an average conductivity is a model approximation requiring its own justification.

Investigation 19.8 (A trial flow gives the opposite bound)**.** Suppose an admissible trial potential has energy A, and an admissible unit flow has energy B. Bound the true capacity.

Solution. Dirichlet's principle gives $C \le A$, while unit-flow duality gives $1/C \le B$. Hence

$$B^{-1} \le C \le A.$$

This produces a rigorous bracket from two independent admissible objects. The lower bound requires exact divergence and flux constraints, not just small sampled residuals. A numerical field that approximately satisfies them needs a certified correction before it is an admissible trial flow.

Chapter 20

Electrostatics and heat: what the equations model

Source correspondence. Lyubich, Remark 10.22, expanded into explicit planar models.

20.1 A line charge in a translationally invariant device

Consider a homogeneous isotropic dielectric with permittivity $\varepsilon > 0$ in SI units $\mathrm{F\,m^{-1}}$. Suppose the geometry, boundary potential, and charge distribution are independent of a third coordinate. The cross-section is a bounded regular planar domain U, and a line charge of density λ in $\mathrm{C\,m^{-1}}$ lies at p. For a grounded conductor at ∂U, the planar model is

$$-\varepsilon \Delta V = \lambda \delta_p, \qquad V|_{\partial U} = 0, \qquad V(z) = \frac{\lambda}{2\pi\varepsilon} G_U(z,p). \tag{36}$$

The units of λ/ε are volts; G is dimensionless. The electric field and electric displacement are

$$E = -\nabla V \quad [\mathrm{V\,m^{-1}}], \qquad D = \varepsilon E \quad [\mathrm{C\,m^{-2}}]. \tag{37}$$

For a circle surrounding the line whose closed interior disk lies in U, the two-dimensional flux $\int D \cdot n \, ds$ is charge per axial length, and equals λ. This is not the potential of a three-dimensional point charge: the latter uses a $1/r$ singularity. The logarithm is a consequence of translation invariance and two-dimensional transverse divergence.

For a smooth grounded boundary, the induced conductor charge per axial length and per boundary arclength is

$$\sigma_{\mathrm{ind}} = \varepsilon \partial_n V = \frac{\lambda}{2\pi} \partial_n G, \qquad \int_{\partial U} \sigma_{\mathrm{ind}} \, ds = -\lambda. \tag{38}$$

To check the sign, the conductor's normal into the field region is $-n$, whereas $E = -\nabla V$. Therefore $\varepsilon E \cdot (-n) = \varepsilon \partial_n V$. For positive λ, the derivative is negative and the induced charge is negative. Combining (13) and (38), its normalized magnitude is exactly harmonic measure:

$$-\sigma_{\mathrm{ind}} \, ds/\lambda = d\omega^p.$$

This is the precise mathematical content behind the electrostatic interpretation. It also identifies which part fails at a rough boundary: the charge measure may remain meaningful while a classical surface density is unavailable.

20.2 A finite-radius source and the logarithmic energy divergence

An ideal line charge has infinite field energy near its axis. In a grounded circular cylinder of outer radius b, regularize the source by a coaxial inner conductor of radius $a > 0$. Outside that conductor,

$$V(r) = \frac{\lambda}{2\pi\varepsilon}\log(b/r), \qquad |E(r)| = \frac{|\lambda|}{2\pi\varepsilon r}, \qquad a < r < b. \tag{39}$$

The voltage difference, capacitance per unit length, and stored energy per unit length are

$$\begin{aligned} V_0 &= \frac{\lambda}{2\pi\varepsilon}\log(b/a), \\ C' &= \frac{\lambda}{V_0} = \frac{2\pi\varepsilon}{\log(b/a)}, \\ W' &= \frac{\varepsilon}{2}\int_{a<|z|<b} |E|^2 dA = \frac{\lambda^2}{4\pi\varepsilon}\log(b/a) = \frac{1}{2}C'V_0^2. \end{aligned} \tag{40}$$

Here C' has units $\mathrm{F\,m^{-1}}$ and W' has units $\mathrm{J\,m^{-1}}$. If charge density λ is fixed, then $W' \to \infty$ as $a \downarrow 0$. If voltage V_0 is fixed instead, then $C' \to 0$, $\lambda = C'V_0 \to 0$, and $W' \to 0$. These two limits hold different physical inputs fixed and must not be confused.

For a noncircular boundary the finite regular part of G changes the finite term in an excised energy; the local logarithmic divergence has the same coefficient. This remains valid for a regular boundary without assuming it has a classical normal derivative. Near the pole write $G = -\log r + h(0) + O(r)$. For a sufficiently small fixed ϵ, choose a regular value $t > 0$ below the minimum of G on the pole circle of radius ϵ. Apply Green's first identity on $\{G > t\} \setminus \overline{D_\epsilon(p)}$. Its analytic outer boundary contributes $-2\pi t$ by flux conservation, whereas the inner circle contributes $2\pi\log(1/\epsilon) + 2\pi h(0) + O(\epsilon|\log\epsilon|)$. Let regular values $t \downarrow 0$. The integration regions increase to $U \setminus \overline{D_\epsilon(p)}$, so monotone convergence for $|\nabla G|^2$ passes the identity to the full domain. Finally letting $\epsilon \downarrow 0$ gives

$$\int_{U\setminus D_\epsilon(p)} |\nabla G|^2 dA = 2\pi\log(1/\epsilon) + 2\pi h(0) + o(1). \tag{41}$$

The regular part and the coordinate length ϵ must be transformed together under a change of scale. Their combination is invariant; either term separately depends on the coordinate convention.

20.2.1 An exact remainder in the excised energy

There is a sharper identity behind the $o(1)$ in (41). Choose a pole coordinate disk on which h is harmonic, and let r be any sufficiently small coordinate radius. Its inner-boundary contribution can be evaluated without estimating each term separately:

$$\begin{aligned} \int_{|w|=r} G\partial_n G\, ds &= \int_{|w|=r} (-\log r + h)(r^{-1} - \partial_r h)\, ds \\ &= 2\pi\log(1/r) + 2\pi h(0) - \int_{|w|=r} h\partial_r h\, ds \\ &= 2\pi\log(1/r) + 2\pi h(0) - \int_{D_r} |\nabla h|^2 dA. \end{aligned}$$

The mean-value identity evaluates $\int h\, ds/r$ as $2\pi h(0)$. The omitted mixed term vanishes exactly because $\int_{|w|=r} \partial_r h\, ds = \int_{D_r} \Delta h\, dA = 0$. The last line is Green's first identity for the nonsingular harmonic function h on the full coordinate disk.

Performing the same regular-level exhaustion as in (41) consequently gives the exact equality

$$\int_{U\setminus D_r} |\nabla G|^2 dA = 2\pi \log(1/r) + 2\pi h(0) - \int_{D_r} |\nabla h|^2 dA.$$

If $|\nabla h| \le A_1$ on that disk, the finite-radius remainder lies between $-\pi r^2 A_1^2$ and zero. This improves the preliminary $O(r|\log r|)$ bound because harmonicity cancels its leading error terms. The gradient energy is conformally invariant, so the calculation uses coordinate area without adding a metric factor.

For a rescaled coordinate $\widetilde{w} = cw$, $c \neq 0$, the new regular part at the pole is $\widetilde{h}(0) = h(0) + \log|c|$, and the same deleted disk has radius $\widetilde{r} = |c|r$. Therefore

$$\log(1/\widetilde{r}) + \widetilde{h}(0) = \log(1/r) + h(0).$$

This explicitly checks the scale cancellation in the finite term. For the physical line source, multiply the entire energy identity by $\lambda^2/(8\pi^2\varepsilon)$, the factor coming from $V = \lambda G/(2\pi\varepsilon)$ and physical energy $\varepsilon \int |\nabla V|^2/2$. The finite-radius remainder is then bounded in magnitude by $\lambda^2 r^2 A_1^2/(8\pi\varepsilon)$ per axial length.

20.3 Steady heat conduction with a resolved source

Let $k > 0$ be constant thermal conductivity in $\mathrm{W\,m^{-1}\,K^{-1}}$. In a geometry invariant along an axis, a source per unit axial length P' has units $\mathrm{W\,m^{-1}}$. For a transverse point idealization and boundary temperature T_b,

$$-k\Delta T = P'\delta_p, \qquad T = T_b + \frac{P'}{2\pi k} G_U(\cdot, p). \tag{42}$$

The temperature rise has units kelvin. A finite-radius heater in a circular enclosure gives

$$T(a) - T_b = \frac{P'}{2\pi k}\log(b/a), \qquad \mathsf{R}'_{\mathrm{th}} = \frac{\log(b/a)}{2\pi k}, \tag{43}$$

where the resistance relates temperature difference to power per length. It is the same logarithmic resistance as the condenser, with the constitutive constant changed.

For a smooth distributed volumetric heat source q in $\mathrm{W\,m^{-3}}$, independent of the axial coordinate, the equation is $-k\Delta T = q$. On an analytic bounded cross-section with zero temperature rise at its boundary, representation (12) gives

$$T(p) - T_b = \frac{1}{2\pi k}\int_U G_U(z, p) q(z)\, dA(z). \tag{44}$$

For $q \ge 0$, positivity of G implies $T \ge T_b$. If $0 \le q \le q_{\max}$ and $U = D_b$, comparison with

$$T_{\max}(r) - T_b = \frac{q_{\max}}{4k}(b^2 - r^2)$$

gives the uniform bound $T - T_b \le q_{\max} b^2/(4k)$. This bound is proved for the constant-coefficient steady model. Radiation, nonlinear conductivity, contact resistance, axial end effects, and time-dependent sources require different equations or additional terms; no assertion about a specific device is made by the bound alone.

20.3.1 Replacing an ideal line heater by a resolved disk

A source-radius dependence can be separated exactly from numerical approximation. In a disk of radius b, distribute a fixed power per length $P' > 0$ uniformly over the concentric radius-a disk, where $0 < a < b$. Then $q = P'/(\pi a^2)$ inside the source and $q = 0$ outside. For constant $k > 0$ and boundary temperature T_b, the exact radial solution is

$$T(r) - T_b = \begin{cases} \dfrac{P'}{2\pi k}\log(b/a) + \dfrac{P'}{4\pi k}(1 - r^2/a^2), & 0 \le r \le a, \\ \dfrac{P'}{2\pi k}\log(b/r), & a \le r \le b. \end{cases}$$

To derive it, integrate $-(rT')' = rq/k$ from zero to $r < a$; boundedness of T' at the center gives $T' = -P'r/(2\pi k a^2)$. Outside the source the equation is $(rT')' = 0$, so conservation of the total power gives $T' = -P'/(2\pi k r)$. The two derivatives agree at $r = a$. Integration from the prescribed boundary value fixes the outer expression, and continuity at a fixes the inner constant. Thus the solution is C^1, piecewise C^2, and satisfies the equation distributionally across the source interface; there is no unintended delta source at $r = a$ because its normal flux is continuous.

The central temperature and total outward flux are

$$T(0) - T_b = \frac{P'}{2\pi k}\log(b/a) + \frac{P'}{4\pi k}, \qquad -2\pi b k T'(b) = P'.$$

At every radius $r \ge a$ the resolved source agrees exactly with the ideal centered line-source potential. Inside the source, the logarithmic idealization diverges while the resolved temperature stays finite. Consequently no uniform error estimate on the whole disk can compare these two models without excluding the singular center or specifying a finite source. Agreement outside the source is an exact radial identity, not evidence that every source shape has the same interior temperature.

For fixed P', b, k, reducing a by a factor $s > 1$ raises the center temperature by $P' \log s/(2\pi k)$. This model sensitivity is separate from a discretization tolerance. It persists even if the displayed piecewise formula is evaluated exactly.

20.4 Transient relaxation on a rectangle

Let $\Omega = (0, A) \times (0, B)$, let mass density be $\rho_m > 0$, and let specific heat be $c_p > 0$. The thermal diffusivity is $\alpha = k/(\rho_m c_p)$ in $\mathrm{m^2\,s^{-1}}$. After subtracting a fixed steady solution, a perturbation v satisfies

$$v_t = \alpha \Delta v, \qquad v|_{\partial\Omega} = 0. \tag{45}$$

For a smooth solution, multiplying by v and integrating gives the exact dissipation identity

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 dA = -\alpha \int_\Omega |\nabla v|^2 dA. \tag{46}$$

One may prove a sufficient Poincaré estimate without a spectral theorem. Since $v(0, y) = 0$,

$$|v(x, y)|^2 = \left|\int_0^x v_x(s, y) ds\right|^2 \le x \int_0^A |v_x(s, y)|^2 ds.$$

Integrating over x and then y yields $\|v\|_2^2 \le (A^2/2)\|v_x\|_2^2$. Repeating in the other direction and adding the two lower bounds gives

$$\|\nabla v\|_2^2 \ge 2(A^{-2} + B^{-2})\|v\|_2^2.$$

Substitution in (46), followed by integration of the scalar differential inequality, gives

$$\|v(t)\|_2 \le e^{-2\alpha(A^{-2}+B^{-2})t}\|v(0)\|_2. \tag{47}$$

This estimate is not sharp, but its constant has been derived. For the explicit mode

$$v(x,y,t) = C\sin(\pi x/A)\sin(\pi y/B)e^{-\alpha\pi^2(A^{-2}+B^{-2})t}, \tag{48}$$

direct differentiation verifies (45) and its exact faster decay. No measured relaxation time is being reported; (47) is an analytic bound and (48) is an exact solution of the stated idealized equation.

20.5 Solved investigations

Investigation 20.1 (A dimensional check on a line source)**.** Why is $\lambda G/(2\pi\varepsilon)$ a voltage, while $\lambda G/(2\pi)$ is not?

Solution. $[\lambda]$ = C/m and $[\varepsilon]$ = F/m = C/(V m). Their quotient has units volts. The dimensionless mathematical kernel accounts for geometry; the constitutive constant converts charge density to voltage. Omitting it changes the modeled quantity, even if the plotted level sets happen to remain the same.

Investigation 20.2 (A source closer to one circular wall point)**.** For the unit disk and real $p = a \in (0,1)$, compare the induced charge density at $\zeta = 1$ and $\zeta = -1$.

Solution. Their normalized magnitudes are proportional to $(1-a^2)/(1-a)^2$ and $(1-a^2)/(1+a)^2$. Their ratio is

$$\frac{|\sigma_{\mathrm{ind}}(1)|}{|\sigma_{\mathrm{ind}}(-1)|} = \left(\frac{1+a}{1-a}\right)^2.$$

For $a = 1/3$ the ratio is 4. This is an exact value for the planar grounded-disk model, not an experimental measurement of charge density.

Investigation 20.3 (Uniform heat generation)**.** Solve $-k\Delta T = q_0$ on a disk of radius b with $T = T_b$ at its boundary, and find its outward heat flux per axial length.

Solution. The solution is $T = T_b + q_0(b^2 - r^2)/(4k)$. Its outward heat flux is $-kT_r = q_0 r/2$, so

$$\int_{|z|=b} (-k\partial_n T)\, ds = \frac{q_0 b}{2}\, 2\pi b = q_0 \pi b^2.$$

This equals the integrated heat generation over the cross-section, providing an exact conservation check independent of the Green integral.

Investigation 20.4 (A guaranteed relaxation time)**.** Use (47) to ensure $\|v(t)\|_2 \le \eta\|v(0)\|_2$ for $0 < \eta < 1$.

Solution. It suffices that

$$t \ge \frac{\log(1/\eta)}{2\alpha(A^{-2}+B^{-2})}.$$

Doubling both side lengths while keeping diffusivity fixed multiplies this sufficient time by four. This scaling follows from the Laplacian's length dimension and remains visible before any numerical simulation.

Chapter 21

Deterministic approximation with explicit error budgets

21.1 What an approximation is required to certify

A numerical value and a theorem about its error are different outputs. Here the target is stated before selecting an algorithm. Given continuous boundary data g on the unit circle and a radius $0 < r_0 < 1$, the first task is to produce a function v and a number B such that

$$\sup_{|z|\le r_0} |v(z) - H_g(z)| \le B, \tag{49}$$

where H_g denotes the exact harmonic extension. A finite grid of small errors is not this assertion: the supremum is over a continuum. The Fourier estimates below prove it uniformly. A later finite-difference certificate first bounds errors at mesh nodes and then adds a separate interpolation bound to obtain a continuum statement.

Every bound has input hypotheses. In particular, a declared bound $\|g\|_\infty \le M$ or $\|u_{xxxx}\|_\infty \le M_4$ is an assumption until it is proved from a formula or independently certified. Differentiating at many sample points does not prove a derivative bound between them. This chapter establishes analytic certification formulas and exact model examples. It does not report software runs, floating-point enclosures, CAS confirmations, or proof-assistant kernel checks.

21.2 Poisson–Fourier truncation on a compact interior disk

For $g \in C(\partial\mathbb{D})$ define complex Fourier coefficients by

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})e^{-in\theta}\,d\theta, \qquad n \in \mathbb{Z}.$$

For real data $c_{-n} = \overline{c_n}$, and $|c_n| \le M := \|g\|_\infty$. Expanding the Poisson kernel by geometric series gives, for $z = re^{i\theta}$,

$$H_g(z) = c_0 + \sum_{n\ge1}\left(c_n z^n + \overline{c_n}\,\overline{z}^{\,n}\right) = c_0 + 2\,\mathrm{Re}\sum_{n\ge1} c_n z^n. \tag{50}$$

The expansion is absolutely and uniformly convergent on $|z| \le r_0$. Indeed, the summands are bounded by $2Mr_0^n$, a summable geometric majorant. This also justifies exchanging the boundary integral with the kernel expansion. No convergence of the unweighted boundary Fourier series is needed.

Theorem 21.1 (Uniform value and gradient tails)**.** *Let* $H_N = c_0 + 2\,\mathrm{Re}\sum_{n=1}^{N} c_n z^n$. *For* $0 < r_0 < 1$,

$$\|H_g - H_N\|_{L^\infty(D_{r_0})} \le \frac{2Mr_0^{N+1}}{1-r_0},$$
$$\|\nabla(H_g - H_N)\|_{L^\infty(D_{r_0})} \le \frac{2Mr_0^{N}((N+1)-Nr_0)}{(1-r_0)^2}. \tag{51}$$

Proof. The first inequality is the geometric tail estimate $2M\sum_{n>N} r_0^n$. For a holomorphic monomial, the Euclidean norm of the gradient of $2\,\mathrm{Re}(c_n z^n)$ is exactly $2n|c_n||z|^{n-1}$. Summing these norms gives

$$2M\sum_{n>N} nr_0^{n-1} = 2M\frac{d}{dr_0}\left(\frac{r_0^{N+1}}{1-r_0}\right),$$

which is the second expression in (51). Differentiation is legitimate on compact interior disks because the differentiated series has its own summable majorant. Taking absolute values before summation makes the bounds independent of cancellations and hence uniform in angle. □

For a desired value-tail tolerance $\tau > 0$, it is sufficient to choose

$$N + 1 \ge \frac{\log(2M/((1-r_0)\tau))}{\log(1/r_0)}, \tag{52}$$

when the numerator is positive; otherwise $N = 0$ already satisfies the tail bound. The denominator tends to zero as $r_0 \uparrow 1$. This quantifies the boundary-layer difficulty of recovering fine boundary features from an interior truncation. A method efficient on a fixed compact disk need not be efficient uniformly up to the boundary.

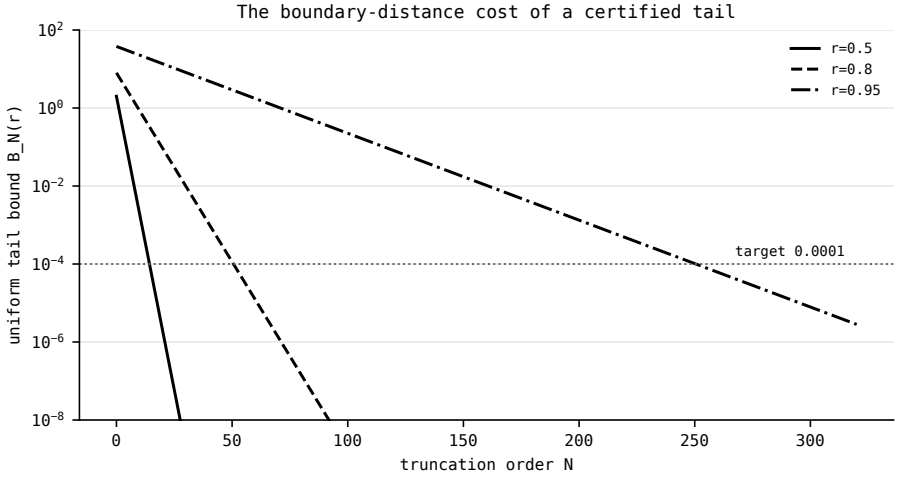


Figure 21.1: Certified value-tail bounds $2r_0^{N+1}/(1-r_0)$ for $M = 1$ and several interior radii. The target line is an error requirement, not a measured residual. The slower decay as r_0 approaches one is a consequence of (51).

21.3 Energy carried by individual Fourier modes

The value estimate (51) has an energy analogue. For the real mode $u_n(z) = 2\,\mathrm{Re}(c_n z^n)$,

$$\int_{D_r} |\nabla u_n|^2 dA = \int_0^r \int_0^{2\pi} 4n^2|c_n|^2 s^{2n-2}\, s\, d\theta\, ds = 4\pi n|c_n|^2 r^{2n}.$$

Distinct modes are orthogonal for the gradient inner product: their cross term contains the angular factor $e^{i(n-m)\theta}$, whose integral is zero when $n \neq m$. Integrating finite partial sums and then using their uniform derivative convergence on D_r, $r < 1$, gives

$$\mathcal{E}_{D_r}(H_g) = 4\pi \sum_{n\geq 1} n|c_n|^2 r^{2n}, \quad \mathcal{E}_{D_r}(H_g - H_N) = 4\pi \sum_{n>N} n|c_n|^2 r^{2n}. \tag{52a}$$

With $q = r^2$ and $|c_n| \leq M$, the latter is bounded by

$$4\pi M^2 \frac{q^{N+1}((N+1) - Nq)}{(1-q)^2}. \tag{52b}$$

The same geometric differentiation used for (51) proves this formula. Letting $r \uparrow 1$ and using monotone convergence shows that the full disk extension has finite energy exactly when $\sum_{n\geq 1} n|c_n|^2 < \infty$. Continuous boundary data alone do not automatically supply that weighted summability. Interior smoothness and finite interior energy therefore coexist with a possible divergence of energy near the boundary.

These identities explain why a small value error and a small energy error are distinct requirements. A boundary mode $a_n \cos(n\theta)$ has maximum size $|a_n|$ but energy $\pi n a_n^2$. Its frequency enters the second quantity. A certificate for electrostatic field energy must control derivatives or this weighted coefficient sum, rather than only the amplitude of an approximate potential.

21.4 Coefficient acquisition and data uncertainty

Suppose that available coefficients $\widehat{c}_n$ satisfy proved absolute error bounds $|\widehat{c}_n - c_n| \leq \delta_n$, and let $\widehat{H}_N = \widehat{c}_0 + 2\,\mathrm{Re}\sum_{n=1}^{N} \widehat{c}_n z^n$, with $\widehat{c}_0$ real. Then the triangle inequality gives the uniform certificate

$$\|\widehat{H}_N - H_g\|_{\infty, D_{r_0}} \leq \frac{2M r_0^{N+1}}{1 - r_0} + \delta_0 + 2\sum_{n=1}^{N} \delta_n r_0^n. \tag{53}$$

An additional rigorously bounded evaluation error η_{arith} can be added to the right side. Printing many decimal places does not provide such an arithmetic bound.

Proposition 21.2 (A deterministic quadrature certificate)**.** *Suppose $\theta \mapsto g(e^{i\theta})$ is periodic and Lipschitz with constant L, and $|g| \leq M$. The equally spaced left-endpoint rule*

$$\widehat{c}_n = \frac{1}{m} \sum_{j=0}^{m-1} g(e^{2\pi i j/m}) e^{-2\pi i n j/m}$$

satisfies, in exact arithmetic,

$$|\widehat{c}_n - c_n| \leq \frac{\pi}{m}(L + |n|M). \tag{54}$$

Proof. The function $f_n(\theta) = g(e^{i\theta})e^{-in\theta}$ has Lipschitz constant at most $L + |n|M$. This follows by adding and subtracting $g(e^{is})e^{-int}$ and using $|e^{-int} - e^{-ins}| \leq |n||t - s|$. On a subinterval of length $h = 2\pi/m$, the absolute error between the integral of f_n and its left-endpoint rectangle is at most

$$\int_0^h (L + |n|M)s\, ds = \frac{1}{2}(L + |n|M)h^2.$$

Summing m such errors and dividing by 2π gives (54). This argument has no probabilistic sampling assumption and does not require that the sampled coefficients avoid aliasing; aliasing is included in the deterministic bound. □

Combining (53) and (54), and bounding the finite weighted sums by infinite geometric sums, gives the convenient sufficient estimate

$$B_{N,m}(r_0) = \frac{2Mr_0^{N+1}}{1-r_0} + \frac{\pi}{m}\left(L\frac{1+r_0}{1-r_0} + \frac{2Mr_0}{(1-r_0)^2}\right). \tag{55}$$

This bound may be conservative, but every term has a defined origin. If each sample itself has error at most η, then each coefficient has an additional error at most η in exact summation. The resulting additional value bound is at most

$$\eta\left(1 + 2\sum_{n=1}^{N} r_0^n\right) \le \eta\frac{1+r_0}{1-r_0}. \tag{56}$$

This coefficient-based bound allows adversarial errors; it is not a variance estimate. If instead a continuous approximate boundary function $\widetilde{g}$ is known with $\|g - \widetilde{g}\|_\infty \le \eta$, the maximum principle gives the sharper contraction

$$\|H_g - H_{\widetilde{g}}\|_{\infty,\mathbb{D}} \le \eta.$$

The two bounds correspond to different available information.

21.5 Arithmetic is a separately bounded operation

The polynomial in (50) may be evaluated by Horner's rule. To state a certificate without making assumptions about a particular processor, suppose that each step is computed with an absolute local error bound ϵ_k. Write the exact recurrence

$$b_N = c_N, \qquad b_k = c_k + zb_{k+1} \quad (k = N-1, \dots, 1), \qquad P(z) = zb_1.$$

If the implemented recurrence gives $\widehat{b}_k$ with

$$|\widehat{b}_k - (c_k + z\widehat{b}_{k+1})| \le \epsilon_k,$$

and $|\widehat{b}_N - b_N| \le \epsilon_N$, induction gives

$$|\widehat{b}_k - b_k| \le \sum_{j=k}^{N} |z|^{j-k}\epsilon_j.$$

If the final multiplication has absolute error ϵ_*, then

$$|\widehat{P} - P| \le \epsilon_* + \sum_{j=1}^{N} |z|^j \epsilon_j. \tag{57}$$

Taking twice the real part multiplies this bound by at most two, and the constant coefficient's error is added separately. Directed interval arithmetic could establish the local ϵ_k from exact input intervals. This text proves the propagation rule; it does not claim that such interval computations have been executed.

With coefficients already available, each query needs $O(N)$ arithmetic operations and $O(1)$ working storage beyond the coefficient list. Computing all $N+1$ coefficients by the displayed quadrature uses $O(mN)$ operations directly. For fixed $r_0 < 1$, the tail alone requires $N = O(\log(1/\tau))$ as $\tau \downarrow 0$, while the Lipschitz quadrature certificate (55) can require $m = O(1/\tau)$. These are operation counts under the stated implementation, not measured times and not optimality claims about all possible algorithms.

21.6 A mesh certificate from the discrete maximum principle

Consider the square $\Omega = (0,1)^2$ with mesh width $h = 1/m$, $m \geq 2$. At interior nodes define

$$(L_h v)_{ij} = \frac{4v_{ij} - v_{i+1,j} - v_{i-1,j} - v_{i,j+1} - v_{i,j-1}}{h^2}. \tag{58}$$

This is the discrete version of $-\Delta$. If $L_h v \geq 0$ at every interior node and $v \geq 0$ at every boundary node, then $v \geq 0$ everywhere. Indeed, if a negative minimum occurs at an interior node, each neighbor is at least as large, so $L_h v \leq 0$ there. Equality is forced and all four neighbors share the same minimum. Repeating along a nearest-neighbor path reaches the boundary, contradicting its nonnegative values. This finite propagation argument proves the comparison principle without spectral information.

The quadratic barrier

$$b(x,y) = \frac{x(1-x) + y(1-y)}{4}$$

satisfies $L_h b = 1$ exactly at all interior nodes, $b \geq 0$ on the boundary, and $\max b = 1/8$. Thus if e has boundary values bounded in absolute value by η and $|L_h e| \leq \delta$, comparison with $\eta + \delta b$ gives

$$\max_{\text{nodes}} |e| \leq \eta + \frac{\delta}{8}. \tag{59}$$

The coefficient $1/8$ is a proved upper bound for the discrete inverse in the maximum norm; it is not an estimated condition number.

21.6.1 Existence of the discrete solve and its comparison operator

The comparison principle also proves that the discrete problem has a unique solution for every prescribed boundary array and interior source. Move the known boundary-neighbor terms to the right side. The resulting linear map acts on the $(m-1)^2$ interior values. If its output is zero and its boundary values are zero, comparison applied to both signs forces the array to be zero. The square matrix therefore has trivial kernel and is invertible. This finite-dimensional argument proves existence as well as uniqueness; convergence of a particular iteration is not needed to define the exact discrete solution.

For two sources $f_1 \leq f_2$ with the same boundary values, their solutions satisfy $L_h(u_2 - u_1) = f_2 - f_1 \geq 0$, and hence $u_1 \leq u_2$. Thus the inverse on zero-boundary arrays preserves order. If r is any signed residual and $e = L_h^{-1} r$, comparison with $\|r\|_\infty b$ yields

$$-\|r\|_\infty b \leq e \leq \|r\|_\infty b, \qquad \|L_h^{-1} r\|_\infty \leq \tfrac{1}{8}\|r\|_\infty.$$

For nonzero boundary error, first subtract its discrete harmonic extension. That extension has norm at most the boundary norm by the same principle. This proves the additive form of (59) directly as the sum of a boundary contribution and a forced contribution.

Theorem 21.3 (Residual plus consistency certificate)**.** *Suppose $u \in C^4(\overline{\Omega})$ solves $-\Delta u = f$. Let v be any nodal array whose boundary error is at most η and whose exact nodal residual satisfies $|L_h v - f| \leq \delta$. Define*

$$M_4 = \|u_{xxxx}\|_\infty + \|u_{yyyy}\|_\infty.$$

Then

$$\max_{\text{nodes}} |v - u| \leq \eta + \frac{\delta}{8} + \frac{h^2 M_4}{96}. \tag{60}$$

If $I_h v$ is bilinear interpolation and $M_2 = \|u_{xx}\|_\infty + \|u_{yy}\|_\infty$, then

$$\|I_h v - u\|_{\infty,\overline{\Omega}} \leq \eta + \frac{\delta}{8} + h^2 \left(\frac{M_4}{96} + \frac{M_2}{8} \right). \tag{61}$$

Proof. Taylor's formula with integral remainder in one coordinate gives

$$\left|\frac{u(x+h)-2u(x)+u(x-h)}{h^2}-u_{xx}(x)\right|\le\frac{h^2}{12}\|u_{xxxx}\|_\infty.$$

The constant $1/12$ can be derived without choosing unspecified Taylor points on the two sides. The exact identity

$$\frac{u(x+h)-2u(x)+u(x-h)}{h^2}=\frac{1}{h^2}\int_{-h}^{h}(h-|s|)u_{xx}(x+s)\,ds$$

follows by integrating twice. Expand $u_{xx}(x+s)$ about $s=0$: its constant term integrates to $u_{xx}(x)$, its linear term integrates to zero by oddness, and its remainder has magnitude at most $s^2\|u_{xxxx}\|_\infty/2$. Therefore the remaining integral is bounded by

$$\frac{\|u_{xxxx}\|_\infty}{2h^2}\int_{-h}^{h}(h-|s|)s^2\,ds=\frac{h^2}{12}\|u_{xxxx}\|_\infty.$$

All derivatives are evaluated on the coordinate segment contained in the closed square. The other coordinate is held fixed, so no mixed fourth derivative enters this stencil estimate. The same estimate in the other coordinate yields $|L_hu-f|\le h^2M_4/12$. Therefore $e=v-u$ satisfies $|L_he|\le\delta+h^2M_4/12$. Applying (59) proves (60).

To pass from nodes to the continuum, bilinear interpolation is a convex combination of the four corner values in each cell, so $\|I_h(v-u)\|_\infty\le\max_{\text{nodes}}|v-u|$. For a one-dimensional interval of length h, linear interpolation has error at most $h^2\|u''\|_\infty/8$: the pointwise remainder is $u''(\xi)(x-a)(x-b)/2$, and $|(x-a)(x-b)|\le h^2/4$. Apply this first in x and then in y. The second interpolation is a positive norm-one averaging operation, so the two errors add to at most $h^2M_2/8$. The triangle inequality now gives (61). In operator notation, write I_x and I_y for the two one-dimensional linear interpolations on a cell. They commute, and

$$u-I_xI_yu=(u-I_xu)+I_x(u-I_yu),\qquad\|I_xv\|_\infty\le\|v\|_\infty.$$

This decomposition explains why the two pure second-derivative bounds add without any mixed-derivative assumption. At cell edges the adjacent formulas coincide, so the cellwise estimate controls the continuous piecewise bilinear function on the whole closed square. □

The theorem separates four logical requirements: boundary accuracy, an independently evaluated residual, a derivative bound for consistency, and a derivative bound for interpolation. Removing any one changes the claim. In particular, a small residual controls the discrete solve; it does not eliminate discretization error, and a nodal certificate does not automatically control values between nodes.

21.6.2 From a computed residual to the exact residual hypothesis

Formula (61) is stated for an exact residual. To connect it to supplied numbers, let v denote the actual nodal array to be certified, let $\widehat{v}$ be supplied values with $|v_{ij}-\widehat{v}_{ij}|\le\epsilon_v$, and suppose supplied source values satisfy $|\widehat{f}_{ij}-f_{ij}|\le\epsilon_f$. If an evaluated residual $\widehat{r}$ obeys the arithmetic enclosure

$$|\widehat{r}_{ij}-(L_h\widehat{v}-\widehat{f})_{ij}|\le\epsilon_{\text{eval}},$$

then the stencil's coefficient sum gives

$$|(L_hv-f)_{ij}|\le|\widehat{r}_{ij}|+\epsilon_{\text{eval}}+\epsilon_f+\frac{8\epsilon_v}{h^2}.$$

The factor eight is 4+1+1+1+1, including the central coefficient. It is a worst-case absolute-error bound, with no cancellation or independence assumption. If stored binary floating-point entries are regarded as exact real inputs, their representation contributes $\epsilon_v = 0$; arithmetic in evaluating the stencil still requires its own enclosure. If instead the entries approximate uncertain data, their uncertainty must be propagated as displayed.

With a certified boundary error η, insert this upper bound for δ into (61). If producing the final interpolated output incurs an additional uniform error ϵ_{out}, the full continuum bound is

$$\|\widehat{I}_h v - u\|_\infty \le \eta + \frac{\|\widehat{r}\|_\infty + \epsilon_{\text{eval}} + \epsilon_f}{8} + \frac{\epsilon_v}{h^2} + h^2\left(\frac{M_4}{96} + \frac{M_2}{8}\right) + \epsilon_{\text{out}}.$$

The apparent amplification ϵ_v/h^2 is conservative because it ignores any structure of the input error. It nevertheless shows why blindly refining a mesh while keeping an uncontrolled arithmetic or input error fixed does not establish convergence. Every term has a separate origin and an explicit norm.

21.7 A proved iteration count for a simple solver

For completeness, consider exact-arithmetic Jacobi iteration for (58) with fixed boundary values. Each updated interior value is one quarter of its four old neighbors plus $h^2 f_{ij}/4$. If $e^{(k)}$ denotes the difference from the exact discrete solution, its error recurrence is

$$e_{ij}^{(k+1)} = \tfrac{1}{4}(e_{i+1,j}^{(k)} + e_{i-1,j}^{(k)} + e_{i,j+1}^{(k)} + e_{i,j-1}^{(k)}). \tag{62}$$

The sine vectors

$$s_{ab}(i,j) = \sin(\pi a i/m)\sin(\pi b j/m), \qquad 1 \le a, b \le m-1,$$

form an orthogonal basis of interior arrays. In one dimension the orthogonality follows by writing products of sines as differences of cosines and summing the finite geometric series; the two-dimensional sum factors into those one-dimensional sums. The elementary identity $\sin(t+q) + \sin(t-q) = 2\sin t\cos q$ shows that their eigenvalues are

$$\mu_{ab} = \frac{1}{2}(\cos(\pi a/m) + \cos(\pi b/m)).$$

Therefore, in the unweighted Euclidean norm of nodal arrays,

$$\|e^{(k)}\|_2 \le \rho_m^k \|e^{(0)}\|_2, \qquad \rho_m = \cos(\pi/m) < 1. \tag{63}$$

For $m \ge 2$, concavity of sine on $[0, \pi/2]$ gives $\sin(\pi/(2m)) \ge 1/m$, and hence

$$1 - \rho_m = 2\sin^2(\pi/(2m)) \ge 2/m^2, \qquad \rho_m^k \le e^{-2k/m^2}. \tag{64}$$

Consequently $k \ge (m^2/2)\log(E_0/\tau)$ suffices for a norm error at most τ if $\|e^{(0)}\|_2 \le E_0$ and $E_0 > \tau$. The special mesh $m = 2$ has one interior node and is solved in one step. Each sweep takes $O(m^2)$ operations, so this elementary certified iteration has cost $O(m^4 \log(E_0/\tau))$. Faster solvers exist, but that fact does not alter the proved count for this particular method.

An a priori E_0 can also be bounded. If the initial interior array is zero, the discrete solution has sup-norm at most $\|g\|_{\infty,\partial\Omega} + \|f\|_{\infty,\text{nodes}}/8$ by the barrier argument. Since there are $(m-1)^2$ interior nodes, its unweighted Euclidean norm is at most $(m-1)$ times this sup-norm. In practice an independently bounded residual in (59) can provide a sharper stopping certificate than this conservative a priori iteration estimate.

21.7.1 An exact manufactured mode separates every error source

Consider the continuum solution and source

$$u(x,y) = \sin(\pi x)\sin(\pi y), \qquad f(x,y) = 2\pi^2 u(x,y), \qquad u|_{\partial\Omega} = 0.$$

This is an analytic control problem, not an experimental dataset. Applying the sine addition formula directly to the stencil gives

$$L_h u = \lambda_h u, \qquad \lambda_h = \frac{8}{h^2}\sin^2(\pi h/2).$$

The exact discrete solution for the sampled source is consequently

$$u_h = a_m u \quad \text{at nodes}, \qquad a_m = \frac{2\pi^2}{\lambda_h} = \frac{\pi^2}{4m^2\sin^2(\pi/(2m))} > 1.$$

The strict inequality follows from $\sin s < s$ for $s > 0$. For even m, the central point $(1/2, 1/2)$ is a node and $u = 1$ there, so the exact maximum nodal error is $a_m - 1$. For odd m, the formula remains valid nodewise but the maximum of the sampled u is smaller than one; the plotted equality for the maximum is therefore restricted to even meshes.

Here $M_4 = 2\pi^4$ and $M_2 = 2\pi^2$, because the respective pure derivatives are $\pi^4 u$ and $-\pi^2 u$. The general certificate therefore gives

$$\max_{\text{nodes}} |u_h - u| \le \frac{\pi^4}{48m^2},$$
$$\|I_h u_h - u\|_\infty \le \frac{1}{m^2}\left(\frac{\pi^4}{48} + \frac{\pi^2}{4}\right).$$

Both are consequences of proved derivative bounds over the full square. Expanding the elementary sine formula also gives $a_m - 1 = \pi^2/(12m^2) + O(m^{-4})$, which describes the leading error for this particular mode. The finite-m certificate above remains the inequality used for acceptance; an asymptotic expansion alone would not replace it.

Starting Jacobi iteration at zero keeps every iterate in this one-mode subspace. With $\rho_m = \cos(\pi/m)$, induction in (62) gives

$$v^{(k)} = (1 - \rho_m^k)u_h, \quad v^{(k)} - u_h = -a_m\rho_m^k u, \quad L_h v^{(k)} - f = -2\pi^2\rho_m^k u.$$

For even $m > 2$, the exact discrete maximum error and its residual certificate are, respectively,

$$a_m\rho_m^k, \qquad \frac{\pi^2}{4}\rho_m^k.$$

Their different constants reflect the conservatism of the universal barrier bound, not disagreement between two computations. The error relative to the continuum at nodes is instead $|(1-\rho_m^k)a_m - 1|\max_{\text{nodes}}|u|$; it can temporarily benefit from cancellation between iteration and discretization errors. The safe sum of their bounds does not rely on that cancellation. As $k \to \infty$, the iteration error vanishes but the continuum nodal error tends to $a_m - 1 > 0$. More solver sweeps do not change the mesh equation being solved.

Finally suppose an implemented iteration has additive local errors $d^{(k)}$ in its array update. In the unweighted Euclidean norm, the same symmetric iteration matrix gives

$$\|e^{(k)}\|_2 \le \rho_m^k\|e^{(0)}\|_2 + \sum_{j=0}^{k-1}\rho_m^{k-1-j}\|d^{(j)}\|_2.$$

This follows by repeatedly substituting $e^{(k+1)} = Te^{(k)} + d^{(k)}$ and applying $\|T\|_2 = \rho_m$. If each of the $(m-1)^2$ update entries has error at most ϵ, then $\|d^{(j)}\|_2 \le (m-1)\epsilon$, and the accumulated term is at most $(m-1)\epsilon(1-\rho_m^k)/(1-\rho_m)$. This estimate explains why the exact-arithmetic iteration count and an arithmetic error budget are separate requirements. It proves a propagation rule; the local error bounds remain inputs to be established for an actual implementation.

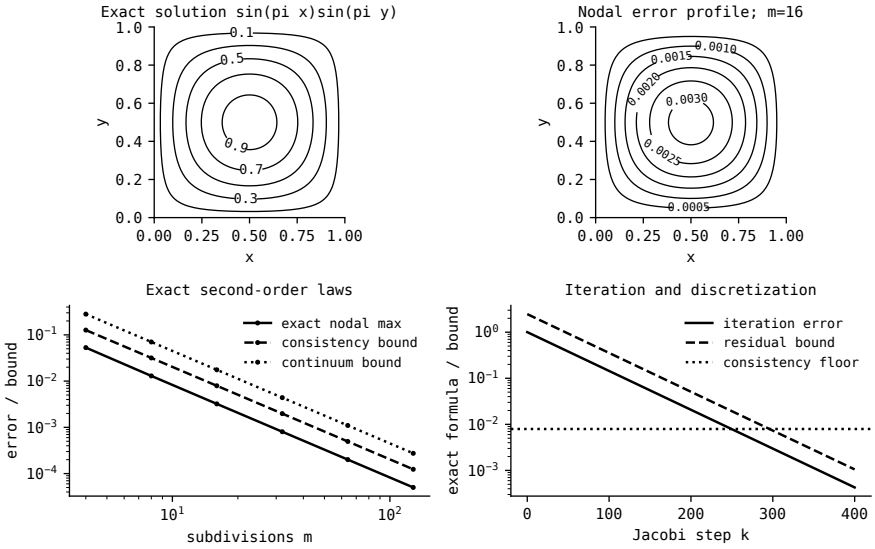


Figure 21.2: An exact manufactured square problem, $u = \sin(\pi x)\sin(\pi y)$ and $f = 2\pi^2 u$. The first panels show the continuum profile and the exact nodal error $(a_m - 1)u$ at $m = 16$. The mesh comparison uses even values $m = 4, 8, 16, 32, 64, 128$, showing the exact nodal maximum, its consistency bound, and the complete continuum bound. At $m = 16$ the final panel shows the exact Jacobi discrete error, its residual certificate, and the fixed consistency bound. The smooth error-profile contours extend the exact nodal formula; they do not depict the error of bilinear interpolation between nodes. Every curve follows from the formulas in this subsection; no measured solver run or CAS check is asserted.

21.8 Solved investigations

Investigation 21.4 (Choosing the truncation degree)**.** With $M = 1$, $r_0 = 1/2$, require a value-tail error at most 10^{-6}. Find a sufficient integer N from (51).

Solution. The bound is 2^{1-N}. Since $2^{20} = 1{,}048{,}576 > 10^6$, $N = 21$ gives $2^{-20} < 10^{-6}$, whereas $N = 20$ gives $2^{-19} > 10^{-6}$. Thus 21 is the smallest integer certified by this particular worst-case tail estimate. It need not be minimal for specific data whose high coefficients are much smaller than one.

Investigation 21.5 (A finite Fourier datum)**.** Let $g(e^{i\theta}) = 2 + 3\cos(2\theta) - \sin(3\theta)$. Find its exact extension and explain the truncation error for $N \geq 3$.

Solution. Its extension is

$$H_g(re^{i\theta}) = 2 + 3r^2\cos(2\theta) - r^3\sin(3\theta).$$

All coefficients of order greater than three vanish, so the truncation error is exactly zero for $N \geq 3$. The generic bound (51) remains a valid positive upper bound but is not sharp. Data-specific coefficient information can replace a uniform worst-case estimate without changing the underlying proof.

Investigation 21.6 (A mesh example with exact consistency)**.** For $u(x, y) = x(1 - x) + y(1 - y)$ on the unit square, determine the consistency error and a continuum interpolation bound.

Solution. Here $-\Delta u = 4$, every pure fourth derivative vanishes, and $L_h u = 4$ exactly. Thus an exact discrete solution with the sampled boundary values agrees with u at every node by

uniqueness. Between nodes, $M_2 = 4$, so (61) gives

$$\|I_h u - u\|_\infty \le h^2/2.$$

At a cell center each one-dimensional concave quadratic has interpolation error $h^2/4$, so the sum attains $h^2/2$. Exact nodal agreement therefore does not imply exact continuum reconstruction.

Investigation 21.7 (Turning a residual into a bound)**.** Suppose a nodal array has exact boundary values and a certified residual $|L_h v - f| \le 8 \times 10^{-7}$. What is its distance from the exact discrete solution, and what is still needed for a PDE claim?

Solution. Formula (59) gives a nodal error at most 10^{-7} relative to the exact discrete solution. A PDE error additionally requires the consistency term involving M_4; a continuum error also requires the interpolation term involving M_2. If the residual was merely computed in ordinary floating point without a rounding enclosure, even the first bound is conditional on the residual being accurate.

Investigation 21.8 (A complete error allocation)**.** Suppose the target is τ, and bounds are available for a Fourier tail, coefficient errors, and polynomial evaluation. Give a sufficient allocation and state how model error enters a physical application.

Solution. Require separately

$$B_{\text{tail}} \le \tau/3, \qquad B_{\text{coeff}} \le \tau/3, \qquad B_{\text{arith}} \le \tau/3.$$

Then (53) and (57) imply a total mathematical error at most τ for the stated boundary-value problem. If the physical quantity differs from the model solution by a separately established bound B_{model}, the physical error is at most $B_{\text{model}} + \tau$, after consistent unit conversion. An unknown model discrepancy cannot be replaced by zero merely because the numerical solve is accurate.

Chapter 22

Source guide and the boundary of the claims

The selected source is Section 10, "Appendix 1: Potential theory," of Mikhail Lyubich's *Conformal Geometry and Dynamics of Quadratic Polynomials*, volumes I–II, in the supplied September 2023 manuscript. It begins on printed page 163 after Exercise 9.30 and ends on page 173 after Exercise 10.24, before the chapter Notes. The selected population comprises 24 numbered units, nine subsection headings and 21 substantive unnumbered blocks. These are source-coverage units, not a count of new theorems or independent discoveries. Section 11 belongs to a later topic and is outside this volume.

The chapters develop the classical material in a different order where the dependencies require it, give detailed proofs and worked investigations, and state corrections to literal source formulas. The source is attributed as the organizing mathematical text. This volume does not reproduce its pages, and the distributed source bundle does not include the external book or its extracted prose. A source correction records a difference in the supplied edition; it is not a claim of novel research.

22.1 Harmonic functions, differentials and basic principles

Remark 10.1: complex differential operators. Chapter 1 fixes the orientation and the convention $*dx = dy$, $*dy = -dx$, so that

$$d^c u = -u_y \,\mathrm{d}x + u_x \,\mathrm{d}y, \qquad dd^c u = (\Delta u)\,\mathrm{d}x \wedge \mathrm{d}y.$$

The source's real-coordinate wedge identity has a printed dz where dx is required. The operator signs are checked from the definitions rather than inferred from a differing convention elsewhere. The unnumbered conjugacy, period and harmonic-one-form statements are included in the same chapter. On a circle of radius r, the period uses $\int_0^{2\pi} r\, u_r \,\mathrm{d}\theta$; the factor r is missing in the source's angular version.

Corollaries 10.2 and 10.3: boundary extrema and uniqueness. The maximum principles in Theorem 2.3 and the boundary-limsup form in Proposition 2.4 give the extrema and uniqueness statements with compact closure and nonempty boundary. A compact surface with empty boundary is excluded: no boundary datum there determines the additive constant. The unnumbered mean-value property is developed both in its smooth Taylor form and in the continuous mean-value characterization.

22.2 Poisson kernels and Harnack compactness

Proposition 10.4: Poisson representation. Theorem 3.1 gives the representation with explicit normalization

$$P(z, e^{it}) = \frac{1 - |z|^2}{|e^{it} - z|^2}, \qquad h(z) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{it}) P(z, e^{it})\, \mathrm{d}t.$$

Boundary-continuous harmonic functions are distinguished from functions harmonic only in the open disk. Derivations using disk automorphisms and the kernel's analytic expression expose the Jacobian and sign conventions.

Proposition 10.5 and Exercise 10.6: disk solvability and the hyperbolic interpretation. Theorem 3.2 proves existence for continuous boundary data by kernel positivity, mass one, interior harmonicity and concentration away from a boundary point. The worked hyperbolic-isometry calculation identifies the disk-to-half-plane map and the pullback of its height function. The three kernel properties listed in the source are separately justified; a boundary limit is not obtained by substituting a boundary point into a singular kernel.

Lemma 10.7: sharp disk Harnack bounds. Theorem 4.1 establishes the factors $(1+r)/(1-r)$ and its reciprocal. For a positive harmonic function without a continuous boundary trace, the proof first uses a smaller circle and passes its radius to one. Thus the use of Poisson representation does not assume nonexistent boundary values.

Theorem 10.8: Harnack chains. The finite-chain argument in Theorem 4.3 gives compact-set comparison from a fixed interior base point on a connected surface domain. Connectedness is required for a chain of coordinate disks; a bound on one component says nothing about an unrelated component.

Proposition 10.9 and Corollary 10.10: normality and increasing limits. Theorem 4.5 uses local uniform absolute bounds, derivative control and compactness. Theorem 4.6 gives the one-point-bounded increasing convergence principle and its infinite alternative. The mean value characterization supplies harmonicity of the limit. A bounded sequence need not itself converge; a monotone sequence has the additional structure needed for that conclusion.

22.3 Subharmonic comparison and logarithmic poles

Definition 10.11 and Remark 10.12: the pointwise convention. Definition 6.1 treats upper semicontinuity, the extended value $-\infty$ and the nontriviality convention explicitly. The volume separates this general class from the finite continuous class used for Perron lifting. Local integrability and the recovery of the canonical pointwise representative are proved in the subharmonic development rather than assumed from an informal picture of logarithmic poles.

Example 10.13: the logarithm. Lemma 8.1 and Corollary 8.2 evaluate the translated circular logarithmic mean, including circles passing through the singularity. The boundary logarithm remains integrable in the tangent case. The unit is an example in the source, not an exercise, and its singularity is handled by an explicit bound rather than by an unspecified continuity assertion.

Remark 10.14: Jensen's formula. Theorem 8.3 includes multiplicities and the contribution of a zero at the center, while distinguishing holomorphic continuation past a circle from merely continuous nonvanishing boundary values. The subsequent Jensen–Riesz development is an extension of the source discussion, with its own hypotheses and measure-theoretic inputs.

Lemma 10.15: harmonic majorization and the converse. The source's first sentence reverses the roles of the two functions. The correct inequality places a subharmonic function below a harmonic one. Theorem 6.4 proves this direction with boundary limsup conditions, and the local comparison characterization uses harmonic functions on each test domain. The unnumbered maximum-principle statement also needs correction: attainment of a global maximum forces constancy, but a flat local plateau need not. The function $\log^+ |z|$ exhibits the difference. Finite maxima, positive sums, conformal invariance, superharmonic duality, pasting and harmonic disk lifting are developed explicitly in Theorems 7.4 and 7.5 and their surrounding results.

22.4 Perron envelopes and regular boundary values

Proposition 10.16: Perron's alternative. Theorem 11.3 supplies a complete one-point contact proof and a dense-set comparison route. A supremum of continuous functions is generally lower semicontinuous, not upper semicontinuous as stated in the source's proof. Disk lifting, monotone harmonic convergence and direct pointwise bounds repair that step without presuming regularity of the envelope. The definition of a Perron family and both of its closure properties are retained.

Example 10.17: smooth boundary points. Proposition 13.1 constructs a strict power barrier in a wedge wider than the physical closed sector, so that its cosine factor stays strictly positive away from the vertex. Corollary 13.2 then proves the one-sided C^1 case. The possible two-sided approach to a slit is treated by the next result instead of assuming an exterior cone that is not present.

Exercise 10.18: Jordan arcs. Theorem 13.3 uses the explicitly declared local Jordan–Schoenflies theorem, Riemann mapping and the Jordan-domain Carathéodory theorem. It constructs separate harmonic functions on the Jordan sides with identical geometric boundary data, allowing their extensions to agree along a two-sided slit. A topological homeomorphism is not treated as a conformal map. This is a proof relative to those named classical inputs, not an independent new proof of the inputs themselves.

Theorem 10.19: boundary attainment. Theorem 12.5 first globalizes a local positive superharmonic barrier, then uses the explicit pair

$$g(a) - \varepsilon - A_\varepsilon B \le H_g \le g(a) + \varepsilon + A_\varepsilon B, \qquad A_\varepsilon \ge 0.$$

This avoids the source proof's potentially negative multiplier, inconsistent pasting radii and boundary-limit signs. Every boundary point is tested in the Perron class, whether or not it is accessible by a selected curve.

Corollary 10.20: the regular-domain Dirichlet problem. Theorem 14.1 proves existence and uniqueness for continuous data on a relatively compact connected domain with nonempty barrier-regular boundary. The extension and restriction operators form an isometric Banach-space correspondence. A converse from solvability to strict barriers is separately qualified by $|\partial U| \ge 2$; on a singleton boundary, all continuous data are constants and do not detect the irregularity of an isolated puncture.

22.5 Harmonic measure and its normalization

Section 10.8 has no numbered source theorem, but its functional and measure claims are substantive coverage units. Theorem 15.1 uses the positive-functional Riesz–Markov representation theorem as a named measure-theoretic input. The evaluation functional is proved positive, linear and of norm one. Thus its representing measure is a probability. In the disk,

$$\mathrm{d}\omega_{\mathbb{D}}^{z}(e^{it}) = P(z, e^{it})\frac{\mathrm{d}t}{2\pi}.$$

The source's displayed density omits $1/(2\pi)$ under its own kernel convention. The clopen-boundary-set statement is proved and extended to harmonicity of mass functions for all Borel sets, with continuous boundary traces asserted only where justified. Conformal covariance, Harnack comparison, nested-domain averaging and bounded Borel extensions are additional written developments; no Brownian simulation or probabilistic construction is used to certify them.

22.6 Green functions, electrostatics and critical topology

Remark 10.21: regularity and the nonregular extension. The statement that $-G$ itself supplies a strict point barrier is not valid under the source's preceding barrier definition, because it vanishes at other boundary points as well. The Green construction here keeps regularity as an explicit hypothesis and does not infer it from that incomplete argument. The source's further assertion about positive capacity and a relaxed Green boundary condition belongs to general Greenian-domain theory. A full nonregular-domain capacity criterion is not established by the regular-domain proofs in this volume and remains a separately identified obligation. The condenser capacities developed later are explicitly defined variational quantities; they are not silently substituted for that missing general theorem.

Remark 10.22: electrostatic interpretation. The normalization in Proposition 16.4 is $-\Delta G = 2\pi\delta_p$ in a planar coordinate. Physical charge, permittivity and conductivity factors are stated in the application chapters. Equipotentials and gradient trajectories are orthogonal away from critical points by their defining differential equations. These are analytic field models and exact identities, not measurements or hardware experiments.

Theorem 10.23: Green existence and uniqueness. Theorem 16.3 constructs the Green function on a relatively compact regular surface domain. It explicitly specifies the continuous subharmonic Perron competitors on the punctured domain, uses the correct seed $\log^{+}(r/|w|)$ and uses signed circular suprema in the comparison estimates. The printed reversed logarithmic ratios and mislabeled outer domains are corrected. Uniqueness follows after removing the bounded harmonic remainder at the pole. Harmonicity is asserted off the pole, not at a point where the value is $+\infty$.

The unnumbered zero-extension assertion is restricted to the ambient surface with the pole removed. On the whole surface the negative point-mass contribution at the pole prevents subharmonicity. This correction is consistent with the positive logarithmic Green convention and with the opposite sign of a subharmonic logarithmic potential.

Exercise 10.24: critical points and simple connectivity. Theorem 18.4 addresses the full regular, relatively compact surface-domain setting. Its two directions use the single-valued exponential of a conjugate on a simply connected domain and normalized gradient flow on compact Green-level slabs. The topology near the pole is checked from its logarithmic expansion.

The separate finite-boundary critical-point count is a further scoped statement, not a replacement for this general equivalence. Critical points always lie away from the pole.

22.7 Unnumbered blocks and evidence levels

The accompanying source inventory gives individual identifiers to the following 21 blocks, so that an audit does not count only the numbered statements:

1. harmonicity, Cauchy–Riemann equations and local conjugates;
2. periods and monodromy, including the circle normalization;
3. conformal invariance and complex differential operators;
4. closed/co-closed harmonic forms and holomorphic differentials;
5. the mean-value equivalence;
6. harmonic extrema principles;
7. the three Poisson-kernel properties and normalized weak convergence;
8. the subharmonic cone and finite maxima;
9. the corrected subharmonic maximum principle;
10. conformal invariance of subharmonicity;
11. harmonic modification and the seam argument;
12. superharmonic duality;
13. the two closure conditions for a Perron family;
14. the boundary Perron class and its finite bounds;
15. strict local barriers and regularity;
16. the Dirichlet Banach-space correspondence;
17. positive-functional harmonic measure and its normalization;
18. masses of clopen boundary components;
19. the Green definition and the disk model;
20. equipotentials, rays and their critical singularities;
21. zero extension away from the Green pole.

The mathematical evidence in this manuscript consists of written arguments, explicit analytic examples and independently reviewed derivations. The named classical dependencies remain dependencies. Typesetting, archive integrity and figure checks establish properties of the delivered files; they do not turn a mathematical statement into a proof-assistant theorem. New Lean, Rocq and CAS formalization is deferred under the standing content-first instruction. The manuscript makes no claim that the general nonregular capacity criterion, all measure theory, uniformization from first principles, institution-specific syllabus mapping or a hardware-validated physics simulation has been completed.

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